Necessary Optimality Conditions of First and Second Orders in the Problem of Control of Difference Analogy of Barbashin Type Integro-Differential Equation

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***Abstract*— An optimal control problem described by difference analogy of Barbashin type integro-differential equation is considered. Necessary optimality conditions of first and second orders are proved.**

***Keywords*— Barbashin type, difference analogy, necessary optimality conditions, conjugated variables.**

I. Introduction

In our prevous papers [6] we have studied optimal control problems described by a system of Barbashin type integro-differential equations.

In this paper, we study an optimal control problem described by difference analogy of Barbashin type integro-differential equations. Necessary optimality conditions of first and second orders are obtained.

II. Statement of Problem

Assume that the admissible control in the "discrete rectangle"

$$D = \{(t,x):\ t = t_0, t_0+1,\dots,t_1;\ x = x_0, x_0+1,\dots,x_1\}$$

is described by the system of difference equations:

$$z(t+1,x) = \\ = \sum_{s=x_0}^{x_1} K(t,x,s,z(t,s)) + f(t,x,z(t,x),u(t)), \quad (1)$$

with the initial condition

$$z(t_0,x) = a(x), \quad x = x_0, x_0+1,\dots,x_1. \quad (2)$$

Here $K(t,x,s,z)$, $f(t,x,z,u)$ are the given n-dimensional vector-functions continuous in totality of variables together with partial derivatives with respect to z, (z,u) respectively, t_0, t_1, x_0, x_1 are the given numbers, moreover $t_1 - t_0$, $x_1 - x_0$ are natural numbers, $a(x)$ is a given n-dimensional discrete vector-function, $u(t)$ is an r-dimensional discrete vector of control actions with values from the given non-empty, bounded and open set U, i.e.

$$u(t) \in U \subset R^r, \quad t = t_0, t_0+1,\dots,t_1-1. \quad (3)$$

Such control functions are called admissible controls.

On the solutions of problem (1)-(2) generated by all possible admissible controls we determine the functional

$$S(u) = \sum_{x=x_0}^{x_1} \varphi(z(t_1,x)). \quad (4)$$

Here $\varphi(z)$ is the given twice continuously differentiable scalar function.

The admissible control $u(t)$ giving minimum to functional (4) under constrains (1)-(3) is called an optimal control, the appropriate process $(u(t), z(t,x))$ an optimal process.

III. Necessary Optimality Conditions

Assuming $(u(t), z(t,x))$ a fixed admissible process, we introduce the denotation

$$H(t,s,z,u,\psi) = \psi' f(t,x,z,u),$$
$$H_u[t] = H_u(t,x,z(t,x),u(t),\psi(t)),$$
$$H_z[t] = H_z(t,x,z(t,x),u(t),\psi(t)),$$
$$H_{uu}[t] = H_{uu}(t,x,z(t,x),u(t),\psi(t)),$$
$$H_{zz}[t] = H_{zz}(t,x,z(t,x),u(t),\psi(t)),$$
$$H_{uz}[t] = H_{uz}(t,x,z(t,x),u(t),\psi(t)).$$

Here $\psi = \psi(t,x)$ is an n-dimensional vector-function of conjugated variables being a solution of the conjugated problem

$$\psi(t-1,x)=\sum_{s=x_0}^{x_1}\frac{\partial K'(t,s,x,z(t,x))}{\partial z}\psi(t,s)+ +\frac{\partial f'(t,x,z(t,x),u(t))}{\partial z}\psi(t,x), \tag{5}$$

$$\psi(t_1-1,x)=-\frac{\partial\varphi(z(t_1,x))}{\partial z}. \tag{6}$$

Let $\delta u(t)\in R^r$, $t=t_0,t_0+1,...,t_1-1$ be an arbitrary r-dimensional bounded vector-function (admissible variation control). Denote by $\delta z(t,x)$ the solution of the following linearized problem

$$\delta z(t+1,x)=\sum_{s=x_0}^{x_1}\frac{\partial K(t,x,s,z(t,s))}{\partial z}\delta z(t,s)+ +\frac{\partial f'(t,x,z(t,x),u(t))}{\partial z}\delta z(t,x)+ +\frac{\partial f'(t,x,z(t,x),u(t))}{\partial u}\delta u(t), \tag{7}$$

$$\delta z(t_0,x)=0. \tag{8}$$

The solution $\delta z(t,x)$ of problem (7)-(8) is called a state variation, equation (7) is called, an equation in variation for problem (1)-(4).

Applying the development scheme in $[5]$, it is proved that the first and second variations of the quality functional (4) have the forms

$$\delta^1 S(u;\delta u)=-\sum_{t-t_0}^{t_1-1}\sum_{x=x_0}^{x_1}H'_u(t,x)\delta u(t),$$

$$\delta^2 S(u;\delta u)= =\sum_{x=x_0}^{x_1}\delta z'(t_1,x)\frac{\partial^2\ (z(t_1,x))}{\partial z^2}\delta z(t_1,x)- -\sum_{t-t_0}^{t_1-1}\sum_{x=x_0}^{x_1}[\delta z'(t,x)H_{zz}(t,x)\delta z(t,x)+ +2\delta u'(t)H_{zz}(t,x)\delta z(t,x)+\delta u'(t)H_{uu}(t,x)\delta u(t)]. \tag{9}$$

From the classic theory of variational calculus it is clear that along the optimal process $(u(t),z(t,x))$ the first variation of the quality functional equals zero, the second - is non-negative.

From the equality to zero of the first variation quality functional it follows:

Theorem 1. If the set U is open, then for optimality of the admissible control $u(t)$ in the considered problem, it is necessary that the relation

$$H_u(\theta,\xi)=0 \tag{10}$$

be fulfilled for all

$$\theta=t_0,t_0+1,...,t_1-1;\ \xi=x_0,x_0+1,...,x_1.$$

Relation (10) is an analogy of the Euler equation for the considered problem.

Each admissible control being a solution of Euler equation is called classical extremals.

Equation (7) is a Barbashin type linear difference equation. It's solution allows the representation[6]:

$$\delta z(t,x)=\sum_{\tau=t_0}^{t_1-1}F(t,\tau,x)f_u(\tau,x)\delta u(\tau)- -\sum_{\tau=t_0}^{t_1-1}\sum_{s=x_0}^{x_1}\left[\sum_{\alpha=\tau+1}^{t-1}R(t,x,\alpha,s)F(\alpha,\tau,s)\right]\times \times f_u(\tau,s)\delta u(\tau). \tag{11}$$

Here $F(t,\tau,x)$ and $R(t,x;\tau,s)$ are $(n\times n)$ matrix functions being the solutions of the following matrix difference equations:

$$R(t+1,x;\tau,s)=\sum_{\beta=x_0}^{x_1}\frac{\partial K(t,x,\beta,z(t,\beta))}{\partial z}R(t,\beta;\tau,s)+ +\frac{\partial f(t,x,z(t,x),u(t))}{\partial z}R(t,x;\tau,s),$$

$$R(t+1,x;t,s)=-\frac{\partial K(t,x,s,z(t,s))}{\partial z},$$

$$F(t,\tau-1,x)=F(t,\tau,x)\frac{\partial f(t,x,z(\tau,x),u(\tau))}{\partial z},$$

$F(t,t-1,x)=E$, (E - is $(n\times n)$ unique matrix)

We can write formula (11) in the form

$$\delta z(t,x)=\sum_{\tau=t_0}^{t-1}\Big[F(t,\tau,x)f_u(\tau,x)+ +\sum_{s=x_0}^{x_1}\sum_{\alpha=\tau+1}^{t-1}R(t,x;\alpha,s)F(\alpha,\tau,s)f_u(\tau,s)\Big]\delta u(\tau). \tag{12}$$

Let by definition

$$L(t,\tau)=F(t,\tau,x)f_u(\tau,x)+ +\sum_{s=x_0}^{x_1}\sum_{\alpha=\tau+1}^{t-1}R(t,x;\alpha,s)F(\alpha,\tau,s)f_u(\tau,s).$$

Then formula (11) is written by the formula

$$\delta z(t,x)=\sum_{\tau=t_0}^{t_1-1}L(t,\tau)\delta u(t). \tag{13}$$

By means of representation (12) it is proved that

$$\delta z'(t_1,x)\frac{\partial^2\phi(z(t_1,x))}{\partial z^2}\delta z(t_1,x)= =\sum_{\tau=t_0}^{t_1-1}\sum_{s=t_0}^{t_1-1}\delta u'(\tau)L'(t,\tau)\frac{\partial^2\phi(z(t_1,x))}{\partial z^2}L(t_1,s),$$

$$\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1}\delta z'(t,x)H_{zz}(t,x)\delta z(t,x)=$$

$$=\sum_{x=x_0}^{x_1}\sum_{t=t_0}^{t_1-1}\left[\sum_{\tau=t_0}^{t-1}L(t,\tau)\delta u(\tau)\right]' H_{zz}(t,x)\cdot\left[\sum_{s=t_0}^{t-1}L(t,s)\delta u(s)\right]=$$

$$=\sum_{\tau=t_0}^{t_1-1}\sum_{s=t_0}^{t_1-1}\delta u'(\tau)\left[\sum_{x=x_0}^{x_1-1}\sum_{t=\max(\tau,s)+1}^{t_1-1}L'(t,\tau)H_{zz}(t,x)L(t,s)\right]\delta u(s),$$

$$\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1}\delta u'(t)H_{uz}(t,x)\delta z(t,x)=$$

$$=\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1}\left[\sum_{\tau=t_0}^{t-1}\delta u'(t)H_{uz}(t,x)L(t,\tau)\delta u(\tau)\right].$$

Introduce the matrix function

$$K(\tau,s)=$$

$$=-\sum_{x=x_0}^{x_1}L'(t_1,\tau)\frac{\partial^2\phi(z(t_1,x))}{\partial z^2}L(t_1,s)-$$

$$-\sum_{x=x_0}^{x_1}\sum_{t=\max(\tau,s)+1}^{t_1-1}L'(t,\tau)H_{zz}L(t,s).$$

Then the second variation (9) of the quality case (4) is represented in the form

$$\delta^2 S(u;\delta u)=-\sum_{\tau=t_0}^{t_1-1}\sum_{s=t_0}^{t_1-1}\delta u'(\tau)K(\tau,s)\delta u(s)+$$

$$+2\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1}\left[\sum_{\tau=t_0}^{t-1}\delta u'(t)H_{uz}(t,x)L(t,\tau)\delta u(\tau)\right]+$$

$$+\sum_{t-t_0}^{t_1-1}\sum_{x=x_0}^{x_1}\delta u'(t)H_{uu}(t,x)\delta u(t).$$

Theorem 2. For optimality in the classical extremals $u(t)$ in the considered problem, it is necessary that the inequality

$$\sum_{\tau=t_0}^{t_1-1}\sum_{s=t_0}^{t_1-1}\delta u'(\tau)K(\tau,s)\delta u(s)+$$

$$+2\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1}\left[\sum_{\tau=t_0}^{t-1}\delta u'(t)H_{uz}(t,x)L(t,\tau)\delta u(\tau)\right]+ \tag{14}$$

$$+\sum_{t-t_0}^{t_1-1}\sum_{x=x_0}^{x_1}\delta u'(t)H_{uu}(t,x)\delta u(t)\le 0$$

be fulfilled for all $\delta u(t)\in R^r,\quad t=t_0,t_0+1,\ldots,t_1-1.$

Inequality (14) is a rather general necessary optimality condition of second order.

IV. Conclusions

An optimal control problem described by the difference analogy of Barbashin type integro-differential equation is studied. First and second order necessary optimality conditions are obtained.

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