Problem of Optimal Control of the Processes with Unknown Initial Conditions

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Abstract— **The boundary optimal control problems with unknown initial conditions is investigated in the work. We consider two processes: the heat and wave processes. The formulas for the gradient of the target functional are obtained for the considered problems. These formulas allow to use the numerical methods of the first order to solve the problems.**

Keywords— problem without initial conditions; optimal control problem; heat conduction process; wave process

1. Introduction

One of the most important classes of problems of distribution of boundary regimes is the class of "problems without initial conditions". If control of boundary regimes lasts long enough, then due to the friction inherent in any real physical system, the influence of initial data on the process's behavior subsides with the course of time. Thus we naturally get a problem without initial conditions.

Tikhonov A.N. was the first to study boundary-value problems without initial conditions for parabolic and hyperbolic equations in his work [1]. He gave the method of investigating problems without initial conditions, as well as their first rigorous solution [2]. In the well-known work [3], he investigated uniqueness of the solution to problems without initial conditions as applied to the heat conduction equation (Fourier problems).

In the present work, we investigate an optimal control problem for the heat conduction process without initial conditions; we also investigate an optimal control problem without initial conditions, considering, as an example, the wave process arising in hydrocarbon raw material pipeline transportation systems.

2. The Problem Statement

If the heat process is studied long enough time, then the influence of the initial conditions will not have an impact on the temperature distribution at the observation moment of time. Then finding the solution of the heat equation holding a boundary condition of three type which is given for all $t > -\infty$ is considered in this case.

If the stick is bounded, then the boundary conditions are given on both ends of the stick.

2.1. Let us consider the first type boundary problem for bounded stick:

$$u_t = a^2 u_{xx},\ 0 \le x \le l,\ 0 < t \le T, \tag{1}$$

$$u(0,t) = v_0(t), u(l,t) = v_l(t),\ 0 < t \le T. \tag{2}$$

where $u = u(x,t)$ is the process phase state, determined from the solution to the system (1)–(2) under the corresponding admissible value of the optimizable control vector-function $v = (v_0(t), v_l(t))$. Suppose that there are constraints, on the control vector-functions, of the form

$$\underline{v} \le v(t) \le \overline{v},\quad t \in [0,T], \tag{3}$$

proceeding from technological conditions and technical requirements, where $\underline{v}, \overline{v}$ are given, a is the heat conduction coefficient.

This problem is the specific problem without initial conditions, as well as an influence of initial temperature will be less than impact of other factors, if we multiple repeat temperature action on the surface. i.e. if T is sufficiently large, then from a certain point of time $t_0 > 0$ ($t_0 < T$) on ward, the influence of the initial conditions will not have an impact on the system behavior. Suppose that the process initial state is unknown, but the functions

$$u(x,0) = u_0(x),\ x \in [0,l], \tag{4}$$

determining the process initial state belong to some admissible set of functions: $D = \{u_0(x) : x \in [0,l]\}$, for each of which all the conditions of existence and uniqueness of the solution to the corresponding boundary-value problem are fulfilled.

The objective of the problem is to find such values of the boundary controls $v_0(t), v_l(t), t \in (0,T]$, under which the following functional:

$$J(v) = \frac{1}{mesD} \int_D \int_0^l [u(x,T;v,u_0(x)) - \\ - U(x)]^2 \rho(u_0) dx du_0 \to \min \tag{5}$$

takes its minimal value. The functional (5) determines the mean value of the deviation of the process state at $t = T$ from the given desired state $U(x)$ for all possible initial conditions $u_0(x) \in D$; $\rho(u_0)$ are the density functions of the distribution of the initial values on the set D.

2.2. Let the process be described by the following hyperbolic differential equation [4]:

$$u_{tt} = a^2 u_{xx} - \alpha u_t, (\alpha > 0), 0 < x < l, 0 < t < T \quad (6)$$

$$u(0,t) = v_0(t), t \in [0,T]$$
$$u(l,t) = v_l(t), t \in [0,T] \quad (7)$$

where $u = (x,t)$ is the process phase state, determined from the solution to the system (6)–(7) under the corresponding admissible value of the optimizable control vector-function $v = (v_0(t), v_l(t))$, suppose that there are constraints, proceeding from technological conditions and technical requirements, on the control vector-functions, as an example of the form (3).

The summand αu_t at the right hand of (6) corresponds to the friction which is proportional to velocity. If the wave process (6), (7) is studied long enough, i.e. T is sufficiently large, then from a certain point of time $t_0 > 0$ ($t_0 < T$)on ward, the influence of the initial conditions will not have an impact on the system behavior.

Suppose that the process initial state is unknown, but the functions

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \; x \in [0,l], \quad (8)$$

determining the process initial state be long to some admissible set of pairs of functions $D_1 = \{u_0(x), u_1(x) : x \in [0,l]\}$, for each of which all the conditions of existence and uniqueness of the solution to the corresponding boundary-value problem are fulfilled.

The objective of the problem is to find such values of the boundary controls $v_0(t), v_l(t), t \in (0,T]$, under which the functional, as an example given in the following form:

$$J(v) = \frac{1}{mesD_1} \int_D \int_0^l \big[[u(x,T;v,u_0,u_1) - U_T(x)]^2 + + [u_t(x,T;v,u_0,u_1) - V_T(x)]^2 \big] \rho(u_0)\rho(u_1) dx du_0 du_1 \to \min \quad (9)$$

takes its minimal value. The functional (4) determines the mean value of the deviation of the process state at $t = T$ from the given desired state $(U_T(x), V_T(x))$ for all possible initial conditions $(u_0(x), u_1(x)) \in D_1$, $\rho(u_0), \rho(u_1)$ are the density functions of the distribution of the initial values on the set D_1. The time interval $[t_0, T]$, on which the process state does not depend on the values of the initial conditions given at $t = 0$, plays one of the major roles in investigation of the optimal control and boundary-value problems.

3. The Formulas for the Functional Gradient

We can use the method of variation of the optimizable parameters to obtain the formulas for the gradient of the functional [5].

3.1 Let $\psi(x,t) = \psi(x,t;v,u_0)$ be the solution to the next adjoint initial boundary-value problem:

$$\psi_t + a^2 \psi_{xx} = 0, \; 0 < x < l, 0 < t < T, \quad (10)$$

$$\psi(x,T) = 2[u(x,T) - U(x)], \; x \in (0,l), \quad (11)$$

$$\psi(0,t) = 0, \psi(l,t) = 0 \; t \in [0,T]. \quad (12)$$

Here $u(x,T) = u(x,T;v,u_0)$ is the solution to the initial boundary-value problem (1), (2), (4) under any admissible $v = v(t), u_0 = u_0(x)$.

The formulas for the components of the gradient of the target functional with respect to the control functions $v_0(t), v_l(t)$ are determined in the following form:

$$grad_{v_0(t)} J = \frac{1}{mesD} \int_D \psi_x(0,t) \rho(u_0) du_0, \; t \in [0,T], \quad (13)$$

$$grad_{v_l(t)} J = -\frac{1}{mesD} \int_D \psi_x(l,t) \rho(u_0) du_0, \; t \in [0,T]. \quad (14)$$

3.2 Let $\psi(x,t) = \psi(x,t;v,u_0)$ be the solution to the next adjoint boundary-value problem:

$$\psi_{tt} = a^2 \psi_{xx} + \alpha \psi_t, \; 0 < x < l, 0 < t < T, \quad (15)$$

$$\psi(x,T) = 2[u_t(x,T) - V_T(x)], \; x \in (0,l), \quad (16)$$

$$\psi_t(x,T) = -2[u(x,T) - U_T(x)] + \alpha \psi(x,T), \quad (17)$$

$$\psi(0,t) = 0, \psi(l,t) = 0 \; t \in [0,T]. \quad (18)$$

Here $u(x,T) = u(x,T;v,u_0)$ is the solution to the boundary-value problem (6), (7), (9) under any admissible $v = v(t), u_0 = u_0(x)$.

The formulas for the components of the gradient of the target functional with respect to the control functions $v_0(t), v_l(t)$ are determined in the following form:

$$grad_{v_0(t)} J = \frac{1}{mesD} \int_D \psi_x(0,t) \rho(u_0) du_0, \; t \in [0,T], \quad (19)$$

$$grad_{v_l(t)} J = -\frac{1}{mesD} \int_D \psi_x(l,t) \rho(u_0) du_0, \; t \in [0,T]. \quad (20)$$

4. The Numerical Solution to the Problems

For numerical solution to the considered optimal control problems in distributed systems, we propose to use first order iterative optimization methods based on the application of the analytical formulas derived for the gradient of the target functional with respect to the optimizable functions. For example, we can use the gradient projection methods:

$$v^{k+1} = \Pr_V (v^k - \lambda_k gradJ(v^k)), \; k = 0,1,...$$

or conjugate gradient projection methods [5]. Here $v^0 = [v_0^0(t), v_l^0(t)]$ is some given initial value of the control; $gradJ(v)$ is the gradient of the target functional with respect to the optimizable vector-functions; λ_k is the step of one-dimensional search in the line of the anti-gradient of the target functional; $\Pr_V(\cdot)$ is the projection operator on the admissible set V of controls (this operator has a simple form for the positional constraints (3) [5]).

The results of numerical experiments of the solution to the optimal control problems will be given at the presentation.

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