Ranking Efficient DMUs by Bootstrapping Method

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***Abstract*— In this paper a method for ranking efficient Decision Making Units (DMUs) using Bootstrap has been introduced. The main drawback in existing method is that, DMU which is efficient but is not extreme efficient can not be ranked. The method proposed here is able to rank all kinds of efficient DMUs. The method is used in ranking efficient high schools.**

***Keywords*— data envelopment analysis; decision making unit; efficiency; bootstrapping; ranking;**

I. INTRODUCTION

Data Envelopment Analysis (DEA) is a method to determine the efficiencies of a set of homogenous organization units called Decision Making Units (DMUs). These DMUs utilize multiple inputs to produce multiple outputs and the efficiency is measured by the ratio of multiple outputs to multiple inputs. DEA, that is, a methodology based on linear programming model for assessing the relative efficiency of DMUs is developed by Charenes, Cooper, Rhodes [1] and later extended by Banker, Charnes, Cooper [2]. In DEA, efficiency of each DMU should be assessed as highly as possible by assigning the favorable weights to the inputs and outputs. This leads to the weights assigned to one DMU that is often different from the weights assigned to another DMUs. The main drawback of DEA is that substantial number of DMUs comes out to be efficient, particularly when the number of DMUs is less compared to number of inputs and outputs.

Several authors have proposed methods for ranking the best performers (See Andersen and Peterson (AP) [3], Seiford and Zhu [4], Mehrabian, Alirezai and Jahanshahloo (MAJ) [5]; ,and Jahanshahloo and Memariani (CSW) [6]). In some cases, the AP and MAJ models are infeasible [5, 7]. In addition to this difficulty, the AP model may be unstable [5] because of extreme sensitivity to small variations in the data when some DMUs have relatively small values for some of their inputs. Jahanshahloo and Hosseinzadeh [8] ranked the extreme efficient DMUs in DEA models with constant and variable returns to scale by applying l1-norm. In this paper using bootstrap method for ranking DMUs is proposed.

Bootstrapping is a method for estimating the distribution of statistics that are otherwise difficult or impossible to determine. The general idea behind the bootstrap is to use re-sampling to estimate an empirical distribution for the target statistic (Efron, [9]). The bootstrap has been advocated as a way of `analyzing the sensitivity of measured efficiency scores to the sampling variation' (Simar and Wilson, [10]; Ferrier and Hirschberg, [11]). Bootstrapping, developed by Efron, [9] and Efron and Tibshirani, [12] is based on the idea that where little or nothing is known of the underlying data generating process (d.g.) for a sample of observations, the d.g. can be estimated by using the given sample to generate a set of bootstrap samples by focusing on it.

II. BACKGROUND

One way to calculate this measure of technical efficiency is by the following linear programming problem once for each $DMU_t \quad (t=1,...,n)$:

$$\begin{aligned} &\min\ \theta_t \\ \text{st:}\quad &\lambda.Y \geq y_t \\ &\lambda.X \leq \theta_t.x_t \\ &\textstyle\sum_{j=1}^{n}\lambda_j = 1, \\ &\lambda_j \geq 0,\ j=1,...,n \end{aligned} \tag{1}$$

Where Y is the $n \times s$ matrix of the observed outputs of all DMUs, X is the $n \times m$ matrix of the observed inputs for all DMUs, and λ is a n-dimensional row vector of weights that forms convex combination of observed DMUs relative to which the subject DMU's efficiency is evaluated. The constraint in this problem simply describes the input requirement set as given by the observed data.

The AP [3], CSW [6], NORM1 [8] and MAJ [5] models are as follows, respectively:

AP:

$$\begin{aligned} &\min \theta_o \\ st:\ &\textstyle\sum_{j=1, j\neq o}^{n}\lambda_j x_{ij} \leq \theta_o x_{io} && i=1,...,m \\ &\textstyle\sum_{j=1, j\neq o}^{n}\lambda_j y_{rj} \geq y_{ro} && r=1,...,s \\ &\lambda_j \geq 0 && j=1,...,n,\ j\neq o \end{aligned}$$

CSW:

$$\begin{aligned} &\min \textstyle\sum_{j=1}^{n} z_j \\ &st: \\ &\textstyle\sum_{r=1}^{s}u_r y_{rj} + z_j \sum_{i=1}^{m} v_i x_{ij} - \sum_{i=1}^{m} v_i x_{ij} = 0 && j=1,...,n \\ &\textstyle\sum_{r=1}^{s}u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 && j=1,...,n \\ &z_j \geq 0 && j=1,...,n \\ &v_i \geq \varepsilon && i=1,...,m \end{aligned}$$

$$u_r \geq \varepsilon \qquad r = 1,\dots,s$$

$NORM_1$:

$$\min \sum_{i=1}^{m} |x_i - x_o| + \sum_{r=1}^{s} |y_r - y_{ro}|$$

$$st: \quad \sum_{j=1\, j\neq o}^{n} \lambda_j x_{ij} \leq x_i \qquad i = 1,\dots,m$$

$$\sum_{j=1\, j\neq o}^{n} \lambda_j y_{rj} \geq y_r \qquad r = 1,\dots,s$$

$$x_j \geq 0, y_r \geq 0 \qquad i = 1,\dots,m, r = 1,\dots,s$$

$$\lambda_j \geq 0 \qquad j = 1,\dots,n, j \neq o$$

and MAJ model:

$$\min 1 + w_o$$

$$st: \quad \sum_{j=1\, j\neq o}^{n} \lambda_j x_{ij} \leq x_{io} + w_o \qquad i = 1,\dots,m$$

$$\sum_{j=1\, j\neq o}^{n} \lambda_j y_{rj} \geq y_{ro} \qquad r = 1,\dots,s$$

$$\lambda_j \geq 0 \qquad j = 1,\dots,n, \quad j \neq o$$

Note that a DMU's efficiency is a relative measure. It compares a DMU's performance to the best practice performance implicit in the observed input-output combinations. If different input-output combinations were observed, a DMU's efficiency score would likely change. This idea is the bootstrap performed below.

III. The Bootstrap

The essence of bootstrapping is to use computational power as a substitute for theoretical analysis. In this method, artificial, or pseudo-samples are drawn from the original data; the statistic is recalculated on the basis of each pseudo-sample; the resulting bootstrapped measures are then used to construct a sampling distribution for the statistic of interest. Note that in order for the bootstrap to work, the empirical distribution of the sample must be a good representation of the underlying population distribution that generated the sample in first place.

We use the efficiency scores calculated from the original data to form pseudo-samples of artificial data. Each artificial data set is similar to the original data set in that both follow the same distributions of inefficiency; this assures that the levels of performance within the bootstrapped results are within the realm of observed behavior.

The efficiency measures being considered in this article are input-based measures; the bootstrap is performed over the original efficiency scores. For this reason only the inputs are adjusted in the formation of the pseud-samples. The data in the pseudo-samples thus consist of the original output level for all *n* DMUs, the original input data for the DMU whose efficiency is being calculated, and adjusted input data for the remaining *n-1* DMUs. After forming a pseudo-sample, the efficiency of a DMU's original input vector is then assessed relative to the technology implicit in it. Recalculating a DMU's efficiency relative to a large number of pseudo-samples generates a sampling distribution for the efficiency score.

To perform our analysis, we modify a form of the bootstrap that is commonly used in the analysis of regression equations. In this case we re-sample, with replacement, *n-1* times from a uniform distribution over the set of original efficiency scores, $M^* = \{\theta_1^*,\dots,\theta_n^*\}$, produced by solving equation (1) once for each observation in the original data set. A set of pseudo-efficiency score, $M^b = \{\theta_1^b,\dots,\theta_{n-1}^b\}$, $\theta_j^b \in M^*$, *j=1,…,n-1*, are then used to construct a new reference technology relative to which efficiency is recalculated. Note that only *n-1* pseudo-efficiency scores are drawn; we hold the efficiency of the DMU being assessed constant at its original value. A large number of pseudo-samples, say B, are formed, efficiency is calculated relative to each resulting pseudo-reference technology, and the empirical distribution for the efficiency measure is constructed from the resulting B efficiency scores. Note that a total of $B \times n$ pseudo-reference technologies and bootstrapped efficiency scores are generated in this process (*B* pseudo-samples are generated for each of the *n* observed DMUs in the data set). Specifically, the bootstrap we perform proceeds in four steps:

1) Solve equation (1) once for each DMU to obtain the set of empirical technical efficiency scores, $M^* = \{\theta_1^*,\dots,\theta_n^*\}$, based on the observed input and output data, *X* and *Y*.

2) Adjust the observed matrix of inputs, *X* by the calculated efficiency scores, to get a matrix of efficient inputs, $X^* = D.X$, where *D* is a $n \times n$ diagonal matrix as its elements: $\theta_1^*,\dots,\theta_n^*$ (observed efficiency scores).

3) For each DMU $_t$, *t=1,…,n*:

(i) Draw, with replacement *n-1* efficiency scores from the set M^* to get a pseudo-sample of efficiency scores, $M_t(b) = \{\theta_1^b,\dots,\theta_{t-1}^b,\theta_t^*,\theta_{t+1}^b,\dots,\theta_n^b\}; \theta_j^b \in M^*, j = 1,\dots,t-1,t+1,\dots,n$. Note that the t^{th} DMU's efficiency score is maintained at its original level.

(ii) Construct a new matrix of observed pseudo-inputs as follows:

$$X_t{}^b = [D_t{}^b]^{-1}.X^*$$

where $D_t{}^b$ is a $n \times n$ diagonal matrix containing the bootstrapped efficiency scores $\theta_1^b,\dots,\theta_{t-1}^b$, θ_t^*, $\theta_{t+1}^b,\dots,\theta_n^b$ as its diagonal elements. Note that some of the DMUs in the pseudo-sample will be efficient; others will be inefficient. Further note that the t^{th} DMU's original input vector x_t will be contained in the t^{th} row of $X_t{}^b$.

(iii) Calculate the technical efficiency of the t^{th} DMU relative to the pseudo-technology implicit in $X_t{}^b$ and *Y* by solving the linear program:

$$\min \quad \theta_t{}^b$$

$$st: \quad \lambda.Y \geq y_t$$

$$\lambda.X_t^{\ b} \le \theta_t^{\ b}.x_t$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

$$\lambda_j \ge 0, \quad j = 1,\ldots,n$$

to get the bootstrapped efficiency score θ_t^{*b}.

Repeat steps (i)-(iii) B times to get the set of bootstrapped efficiency scores $\{\theta_t^{*1},\ldots,\theta_t^{*B}\}$ for the t^{th} DMU.

4) Put $\theta_t^{*b} = [\theta_t^{*1} + \ldots + \theta_t^{*B}]/B$, $(t = 1,\ldots,n)$ for each DMU.

By increasing the number of steps (B), distinct means of efficiency scores is obtained and this makes possible the ranking of DMUs.

IV. ILLUSTRATION USING HIGH-SCHOOLS

The data used in this study are based on the data collection from 74 high-schools. The high-schools used four inputs to produce three outputs.

36 high-schools from 74 were found to operate on the best practice frontier ($\theta^* = 1$) [12]. In this paper efficient DMU has been ranked using Bootstrap Method. The main drawback in existing methods for ranking efficient DMU is non-extreme efficient DMU in which including such a DMUs do not alter PPS (Production Possibility Set), and methods can not be used for ranking them. The method is powerful in the sense that the repetition of the procure has no limitation. A set of 74 DMU has been chosen, and BCC input orient Model is used to evaluate the efficiency. It is found 36 of these DMUs are efficient. These efficient DMUs have been ranked using AP, CSW, NORM1, MAJ and our method. The result is shown in table 1 which seems quite satisfactory.

TABLE I. THE RESULT RANKING DMUS .

Efficient DMUs	AP	CSW	NORM 1	MAJ	New Method
3	41	60	21	35	9
5	5	23	3	4	6
7	43	39	29	37	31
16	18	**	26	18	13
23	13	11	23	13	29
24	8	10	11	9	8
25	47	27	36	45	36
26	17	32	17	17	14
27	3	21	1	2	3
30	2	**	8	3	5
31	74	73	32	74	33
32	22	24	33	22	25
33	15	56	18	15	10
34	26	7	35	26	32
37	7	**	10	6	7
38	29	14	28	30	34
39	24	9	27	24	28
41	34	26	15	34	24
42	57	57	16	48	26
45	14	**	20	12	21
46	9	31	19	8	12
47	35	43	34	38	30
49	10	49	9	14	15
51	19	29	14	20	11
53	21	62	24	21	17
56	20	25	31	19	22
57	12	17	6	10	20
60	6	15	4	7	4
62	1	8	2	1	1
63	40	52	25	42	35
66	16	48	22	16	16
67	11	**	7	11	18
69	69	70	13	64	19
70	49	41	12	46	23
71	4	**	5	5	2
74	23	16	30	23	27

V. CONCLUSION

In this paper Bootstrap Method is used to rank efficient DMUs. The result is shown in Table 1. It can be seen that the difference between the results obtained by AP, CSW, NORM1, MAJ and our method is not significant.

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