Polyhedral Methodology of Optimization of Control Processes

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***Abstract*— Evolution of criteria base of problems of optimal control is considered. The singularities of methodology of square-law optimization are analyzed. Retrospective and logic of formation of polyhedral optimizing paradigm in control tasks are shown.**

Keywords*— *quality of control processes; criteria of the optimality; square-law optimization; polyhedral optimization

I. Inconsistency of Paradigm of Linearly Square Optimization of Control Processes

At present the optimizing paradigm of quality of control processes the base of which is expressed by the program manifesto of Ja.Z. Tsypkin: «*To optimize everything that is optimized, and that is not optimized, to make optimized*» is dominated in the theory of automatic systems. In doing so the most popularity beginning with the sixtieth last century has got so-called *the square criterion of quality of control processes* and the corresponding to it *the linearly square problem of control* or *the problem of analytical design of optimal regulators* (ADOR), first introduced and determined in the papers of R.E. Kalman [1] and A.M. Letov [2].

It is necessary to recognize that for rise and development of the square criteria of quality promoted not only requirements of the engineering automation, as the tries to develop the unit theory of synthesis of automatic systems on the basis of the achievements of mathematical optimization theory. The given tries are based on the evident dignities of methodology of square optimization of control processes. They are the following: the outer simplicity, the completeness and analyticity of determination, and also the applicability for the extended class of dynamic objects.

However in spite of the extraordinary popularity and evident dignities the methodology of square optimization of control processes was repeatedly subjected to sharp criticism from the aspect of the leading academics. Really for existing possible interpretations of square criterion of quality it has not the clear physical sense and actually only plays a part of formal instrument permitting to use rather simple mathematical device of the square optimization in the problems of design. The given case essentially limits the applied value of paradigm of linearly square optimization of control processes.

The polyhedral criteria of quality of control processes and respectively *the paradigm of polyhedral optimization of control processes* (for the first time being introduced in the paper [3] and getting the further development in the paper [4]) open the new possibilities for theory and practice of automatic systems.

II. Polyhedral Criteria of Quality of Control Processes

Let us consider class of dynamic objects of control is described by linear vector difference equation of state in the form of

$$\boldsymbol{x}(t+1) = \mathbf{A}(t)\boldsymbol{x}(t) + \mathbf{B}(t)\boldsymbol{u}(t),$$

where $t\in\mathbf{T}$ is discrete time, $\mathbf{T}=[0{:}T-1]\subset\mathbf{Z}_+$ is interval of control, $T\geq 1$ is final (terminal) instant time; $\boldsymbol{x} = \mathrm{col}(x_1, x_2, \ldots, x_n)\in\mathbf{X}=\mathbf{R}^n$ is state vector; $\boldsymbol{u} = \mathrm{col}(u_1, u_2, \ldots, u_r)\in\mathbf{U}=\mathbf{R}^r$ is input vector of controls; $\mathbf{X}$ is state space; $\mathbf{U}$ is space of control inputs; $\mathbf{A}$: $\mathbf{T}\to\mathbf{R}^{n\times n}$ and $\mathbf{B}$: $\mathbf{T}\to\mathbf{R}^{n\times r}$ are functional matrices; $\mathbf{Z}_+$ is nonnegative number set; $\mathbf{R}^i$ is i-dimensional real linear space.

Let us discuss the technology of formalization of polyhedral criteria of quality in control problems with the given goal state of the object $\boldsymbol{x}^*$. First of all let us consider the same design of convex analysis [5], as polyhedral functions and polyhedral norms.

The polyhedral function $f(\boldsymbol{x})$: $\mathbf{X}=\mathbf{R}^n\to\mathbf{R}$ is the function, epigraph of which is convex polyhedron. The most important constructive property of any polyhedral function $f(\boldsymbol{x})$ is the possibility of its disjunctive expansion that is the representation in the form of the function of discrete (or so called pointwise) maximum:

$$f(\boldsymbol{x}) = \bigvee_{i=1}^{N} \varphi_i(\boldsymbol{x}) = \max\{\varphi_1(\boldsymbol{x}), \varphi_2(\boldsymbol{x}),\ldots, \varphi_N(\boldsymbol{x})\},$$

where $\varphi_i(\boldsymbol{x})$ ($i=\overline{1,N}$) are linear base functions:

$$\varphi_i(\boldsymbol{x}) = a_{i0} + \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle,\ a_{i0}\in\mathbf{R},\ \boldsymbol{a}_i\in\mathbf{X},\ i=\overline{1,m}\,.$$

The polyhedral norm is norm being the polyhedral function of coordinates. For example, *cubic* (or *chebyshev*) norm:

$$\|\boldsymbol{x}\|_\infty = \max_{1\leq i\leq n} |\,x_i\,|$$

is relative to class of polyhedral norms.

It is clear that the requirements for dynamic structure of trajectories of movement of control object and also for resources (cost) of controlling actions, necessary for realization of the given movement must be represented by the structure of the criterion of quality.

Let us input values $\varepsilon(t)$, $\Delta\boldsymbol{x}(t)$ and $\Delta\boldsymbol{u}(t)$:

$$\varepsilon(t) = \boldsymbol{x}(t) - \boldsymbol{x}^*,\ \Delta\boldsymbol{x}(t) = \boldsymbol{x}(t+1) - \boldsymbol{x}(t),$$
$$\Delta\boldsymbol{u}(t) = \boldsymbol{u}(t+1) - \boldsymbol{u}(t),$$

characterizing respectively distance of the state object from the goal state, phase speed of object and intensity of controlling action during flowing instant time. Choose some polyhedral norms:

$$H_\varepsilon\colon \mathbf{X}\to\mathbf{R};\ H_{\Delta x}\colon \mathbf{X}\to\mathbf{R};\ H_u\colon \mathbf{U}\to\mathbf{R};\ H_{\Delta u}\colon \mathbf{U}\to\mathbf{R}.$$

Then the quality of control process during flowing instant time we may define by exponents of control exactness and inputs for control having the *polyhedral structure* and representing the combination of values $H_\varepsilon(\varepsilon(t))$, $H_{\Delta x}(\Delta\boldsymbol{x}(t))$, $H_u(\boldsymbol{u}(t))$ and $H_{\Delta u}(\Delta\boldsymbol{u}(t))$.

We shall exemplify the following variants of the given indexes:

- *polyhedral exponents of accuracy of control*:

$$P(t) = \lambda_\varepsilon(t)H_\varepsilon(\varepsilon(t)) + \lambda_{\Delta x}(t)H_{\Delta x}(\Delta\boldsymbol{x}(t));$$
$$P(t) = \max\{\lambda_\varepsilon(t)H_\varepsilon(\varepsilon(t)),\ \lambda_{\Delta x}(t)H_{\Delta x}(\Delta\boldsymbol{x}(t))\},$$

- *polyhedral exponents of inputs for control*:

$$Q(t) = \lambda_u(t)H_u(\boldsymbol{u}(t)) + \lambda_{\Delta u}(t)H_{\Delta u}(\Delta\boldsymbol{u}(t));$$
$$Q(t) = \max\{\lambda_u(t)H_u(\boldsymbol{u}(t)),\ \lambda_{\Delta u}(t)H_{\Delta u}(\Delta\boldsymbol{u}(t))\}.$$

Here $\lambda_\varepsilon(t)\geq 0$, $\lambda_{\Delta x}(t)\geq 0$, $\lambda_u(t)\geq 0$, $\lambda_{\Delta u}(t)\geq 0$ are weight coefficients which in particular may have the form of exponential functions: $C_\nu t^\nu$, $\nu\in\mathbf{Z}_+$, C_ν=const. It follows that polyhedral exponents $P(t)$ and $Q(t)$ are related to instant time t, that is they are point exponents.

From the introduced accurate and resource exponents we may form the different polyhedral criteria of quality of control process, for example, the following:

- polyhedral terminal criterion (*mayer* type):

$$F_M = P(T);$$

- polyhedral integral criteria (*lagrangian* type):

$$F_L = \sum_{t=1}^{T} P(t) + \sum_{t=0}^{T-1} Q(t);$$

$$F_L = \max_t\{P(t),\ t\in\mathbf{T}^+\} + \max_t\{Q(t),\ t\in\mathbf{T}\},\ \mathbf{T}^+=[1:T];$$

$$F_L = \max_t(\{P(t),\ t\in\mathbf{T}^+\}\cap\{Q(t),\ t\in\mathbf{T}\}).$$

- polyhedral mixed criteria (*bolza* type):

$$F_B = F_M + F_L.$$

Appearing values $\boldsymbol{x}(T+1)$ and $\boldsymbol{u}(T)$ in the given criteria not being a part of mathematical model of control object, formally we may define as the following:

$$\boldsymbol{x}(T+1) = \boldsymbol{x}(T),\ \boldsymbol{u}(T) = \boldsymbol{u}(T-1).$$

We shall reduce the general form of one criterion of quality of polyhedral type which is very perspective for problems of optimal stabilization.

Let us assume that the purpose of control is the stabilization of balanced state of object $\boldsymbol{x}^*=0$, and also it is necessary for any (perturbed) starting state $\boldsymbol{x}(0)\equiv\boldsymbol{x}_0\neq 0$ the object was damped in final (terminal) instant time $t=T$: $\boldsymbol{x}(T) = 0$. Let $\boldsymbol{u}(t) = \boldsymbol{u}[t, \boldsymbol{x}_0]$ and $\boldsymbol{x}(t) = \boldsymbol{x}[t, \boldsymbol{x}_0]$ are the flowing values of control and state of controlling object in its motion from the starting state $\boldsymbol{x}_0$, and $\Delta\boldsymbol{u}(t)$ and $\Delta\boldsymbol{x}(t)$ are respectively the speeds of their variation.

In general case quality of control process is bound to characterize the controlling action and the corresponding reaction of output of the object. For estimate of quality of stabilization process it is proposed to use the following (first being introduced in the paper [3]) the *polyhedral functionals of loss*, taking into account the dynamic structure of phase trajectories and controlling actions:

$$F = \max_{0\leq t\leq T-1} Q(\boldsymbol{x}(t), \Delta\boldsymbol{x}(t), \boldsymbol{u}(t), \Delta\boldsymbol{u}(t)).$$

Here $Q(\boldsymbol{x}, \Delta\boldsymbol{x}, \boldsymbol{u}, \Delta\boldsymbol{u})$ is polyhedral exponent of weight loss

$$Q(\boldsymbol{x}, \Delta\boldsymbol{x}, \boldsymbol{u}, \Delta\boldsymbol{u}) = \lambda_1(t)q_1(\boldsymbol{x}(t)) + \lambda_2(t)q_2(\Delta\boldsymbol{x}(t)) +$$
$$+ \lambda_3(t)q_3(\boldsymbol{u}(t)) + \lambda_4(t)q_4(\Delta\boldsymbol{u}(t)),$$

where $q_1\colon \mathbf{R}^n\to\mathbf{R}$, $q_2\colon \mathbf{R}^n\to\mathbf{R}$, $q_3\colon \mathbf{R}^r\to\mathbf{R}$ and $q_4\colon \mathbf{R}^r\to\mathbf{R}$ are some positive homogeneous polyhedral functions, and $\lambda_i(t)\geq 0$ are weight coefficients, and also

$$\lambda_1(t) + \lambda_2(t) + \lambda_3(t) + \lambda_4(t) > 0,\ t = \overline{0, T-1}.$$

III. Polyhedral Chebyshev's Criterion

Let us turn our attention to one classical criterion of quality of processes of stabilization of polyhedral type. Let become stable the goal state of object $\boldsymbol{x}^*=0$. As degree of disturbance of balanced state of object let us take the polyhedral exponent:

$$P(t) = H_x(\boldsymbol{x}(t)) = \|\boldsymbol{x}(t)\|_\infty = \max_{i\in[1:n]}|x_i(t)|.$$

Then the most value of the given exponent in interval of the system functioning shows as criterion of quality of stabilization process:

$$F = \max_{t} \{P(t)\} = \max_{0 \le t \le T-1} \| \boldsymbol{x}(t) \|_{\infty} = \max_{0 \le t \le T-1} \max_{i \in [1:n]} | x_i(t) |.$$

The given criterion has meaning of maximum dynamic mistake (that is maximum amplitude of all totality of variables of state) of stabilization system and is called in literature as criterion of uniform approach, maximum digression and chebyshev's criterion. At first it was introduced into practice of controlled movement estimation by P.L. Chebyshev in 1854 for decision of kinematic problems concerned with control of James Watt steam engine. Late in the thirties – in the early forties last century the given creation was suggested by B.V. Bulgakov for formulation and decision of widely well-known problem about accumulation of disturbances in linear system anticipated the appearance the first optimizing formulation of control problems. In automatics the given creation was introduced in 1953 at the Second All-Union conference on the theory of automatic regulation independently by V.V. Solodovnikov [6] and A.A. Feldbaum [7] as universal exponent of quality (dynamic accuracy) of stabilization systems.

Evidently chebyshev's criterion is the special case of polyhedral criterion introduced above with the following choice of polyhedral function Q(x, Δx, u, Δu):

$$Q(\boldsymbol{x}(t), \Delta\boldsymbol{x}(t), \boldsymbol{u}(t), \Delta\boldsymbol{u}(t)) = \max_{i \in [1:n]} | x_i(t) |,$$

so as

$$q_1(\boldsymbol{x}) = \max_{i \in [1:n]} | x_i(t) |,$$

$$q_2(\Delta\boldsymbol{x}) = q_3(\boldsymbol{u}) = q_4(\Delta\boldsymbol{u}) \equiv 0, \lambda_1(t) \equiv 1.$$

At first the application of chebyshev's criterion of quality for the problems of optimal control was considered in 1956 in the paper of R.E. Bellman, I. Glicksburg, O. Gross. Afterwards the leading academics S.E. Dreyfus, L.W. Neustad, J.B. Pearson, W.A. Porter, R. Kulikowski, J. Warga, E.A. Barbashin, K.A. Lurje, N.N. Krasovsky, A.B. Kurjahsky, Yu.S. Osipov, Ja.Z. Tsypkin, N.N. Moiseev, F.L. Chernousjko, V.A. Yakubovich, R. Gabasov, F.M. Kirillova etc. repeatedly emphasized its main importance for applied control problems.

In spite of clear physical meaning, practical significance and history of the chebyshev's criterion of quality has not the wide application in automatics in the absence of constructive methods of the solution of the optimizing problems provided by him. Some tries of the development of the given criterion of quality were undertook in 1960 in the papers of A.A. Pervozvansky and were concerned with uniform optimization of control systems (see [8]).

IV. PERSPECTIVES OF APPLICATION OF POLYHEDRAL CRITERIA OF QUALITY OF CONTROL PROCESSES

Polyhedral methodology (see [9]) opens the real perspectives for wide application of automatic systems of criterion of quality of chebyshev type in theory and practice.

Really the given methodology is directed to the solution of two cardinal problems of modern automatics connected with the problem of the direct exponents quality of control processes, in the first place, and secondly the problem of direct resource and phase on control processes. The application of the polyhedral criteria of quality leads to the new class of optimization problems that is the problems of polyhedral programming, algorithmization of which is based on powerful device and calculating methods of linear programming.

Polyhedral methodology of formalization of control problems is based on the wide spectrum of criteria of quality of polyhedral structure and also polyhedral phase and resource limits for control processes. The polyhedral approach is effective for the decision of some classical and modern problems of discrete control and observation in conditions full definiteness and uncertainty, in regular, critical and conflict situations, including problems:

- ♦ the limiting speed;
- ♦ the analysis and synthesis by method of polyhedral Lyapunov's functions;
- ♦ the barrier control in critical situations
- ♦ the extreme accumulation of indignations;
- ♦ the guaranteed control in the conditions of the initial polyhedral uncertainty;
- ♦ the robust control in the conditions of the parametrical polyhedral uncertainty;
- ♦ the disputed control by contradictory objects;
- ♦ the identifications of a condition of system and environment in conditions of polyhedral uncertainty of the revolting factors.

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