Definition of the Probability Characteristic of the System from Given Region for Case $(2^+,1^-)$

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Abstract— **In this paper has been studied the process of semi-markovian random walk with jumps and two delaying screens. Tthe Laplace transformation of the distribution of a random variable $\tau(\omega)$ is obtained.**

Keywords— process of semi-markovian random walk; the probability space; Laplas transformation

I. INTRODUCTION

Investigation of the ergodic distribution for semi-markovian random walk take process a special place in the theory of random processes. In 1975 V.Smit has proved the ergodic theorem for semi-markovian processes [1]. The general theorem about ergodic for processes with discrete intervention is proved in [2]. In [3] the ergodic theorem for complex semi-markovian processes with delaying screen is proved.

In [4] to find the Laplace transformation of the distribution for case $(1^+,1^-)$ of a random variable $\tau(\omega)$.

II. THE PROCESS CONSTRUCTION

Let the sequence $\{\xi_k,\eta_k\}_{k=\overline{1,\infty}}$, where $\xi_k,\ \eta_k,\ k=\overline{1,\infty}$, are independent identically distributed random variables and independent themselves, $\xi_k>0$ is given on the probability space $(\Omega,\ \Im,\ \mathrm{P}(.))$.

We construct the process [5]

$$X_1(t)=\sum_{k=0}^{m-1}\eta_k,\ \text{if}\ \sum_{k=0}^{m-1}\xi_k\le t<\sum_{k=0}^{m}\xi_k,\qquad m=\overline{1,\infty}$$

where $\eta_0=z\ge 0,\ \xi_0=0$.

We delay process $X_1(t)$ with screen in the zero (see [1]):

$$X_2(t)=X_1(t)-\inf_{0\le s\le 1}(0,X_1(s))$$

Then we delay this process with screen in $a(a>0)$:

$$X(t)=X_2(t)-\sup_{0\le s\le 1}(0,\ X_2(s)-a).$$

This process is called the process of semimarkov random walk with double delaying screens in the "a "end zero.

We introduce a random variable $\tau(\omega)$, meaning the duration of the time in which process $X(t)$ is in a region $(0,a)$.

III. THE FINDING OF THE LAPLACE TRANSFORMATION OF THE DISTRIBUTION OF A RANDOM VARIABLE $\tau(\omega)$

The purpose in this paper to find the Laplace transformation of the distribution of a random variable $\tau(\omega)$. We denote

$$K(t|z)=\mathrm{P}\{\tau>t|X(0)=z\}.$$

It is obvious, that

$$K(t|z)=\mathrm{P}\{\inf_{0\le s\le t}X(s)>0;\ \sup_{0\le s\le t}X(s)<a|X(0)=z\}$$

On total probability form we have:

$$\begin{aligned}K(t|z)&=\mathrm{P}\{\inf_{0\le s\le t}X(s)>0;\ \sup_{0\le s\le t}X(s)<a;\xi_1>t|X(0)=z\}+\\&+\mathrm{P}\{\inf_{0\le s\le t}X(s)>0;\ \sup_{0\le s\le t}X(s)<a;\xi_1<t|X(0)=z\}=\\&=\mathrm{P}\{\xi_1(\omega)>t\}+\\&+\int_{s=0}^{t}\int_{y=0}^{a}\mathrm{P}\{\xi_1(\omega)\in ds;\ z+\eta_1\in dy\}\mathrm{P}\{\tau>t-s|X(0)=y\}\end{aligned}\qquad(1)$$

Then the equation (1) will be written in the following form:

$$\begin{aligned}K(t|z)&=\mathrm{P}\{\xi_1(\omega)>t\}+\\&+\int_{s=0}^{t}\int_{y=0}^{a}\mathrm{P}\{\xi_1(\omega)\in ds\}d_y\mathrm{P}\{\eta_1<y-z\}K(t-s|y)\end{aligned}\qquad(2)$$

Let's denote:

$$\tilde{K}(\theta|z)=\int_{t=0}^{\infty}e^{-\theta t}K(t|z)dt,\qquad \theta>0.\qquad(3)$$

$$\varphi(\theta)=Ee^{-\theta\xi_1},\ \theta>0.$$

If to apply the Laplace transformation on both sides of the equation (1) with respect to t :

$$\int_{t=0}^{\infty} e^{-\theta t}\tilde{K}(t|z)dt = \int_{t=0}^{\infty} e^{-\theta t}\mathrm{P}\{\xi_1(\omega) > t\}dt +$$
$$+\int_{y=0}^{a} d_y \mathrm{P}\{\eta_1 < y - z\}\int_{t=0}^{\infty} e^{-\theta t}\int_{s=0}^{t}\mathrm{P}\{\xi_1(\omega)\in ds\}K(t-s|y) =$$
$$=\frac{1-\varphi(\theta)}{\theta} + \varphi(\theta)\int_{y=0}^{a}\tilde{K}(\theta|y)d_y\mathrm{P}\{\eta_1 < y - z\},$$

then we have the following equation for $\tilde{K}(\theta|z)$:

$$\tilde{K}(\theta|z) = \frac{1-\varphi(\theta)}{\theta} + \varphi(\theta)\int_{y=0}^{a}\tilde{K}(\theta|y)d_y\mathrm{P}\{\eta_1 < y - z\}. \qquad (4)$$

Let's solve this equation in the class for the Laplace distributions. For example, let

$$\eta_1 = \eta_1^+ + \eta_2^+ - \eta_1^-,$$

$$F\{\eta_1 < t\} = \begin{cases} \dfrac{\lambda^2}{(\lambda+\mu)^2}e^{\mu t}, & t < 0, \\ 1 - \dfrac{\mu}{\lambda+\mu}[1+\dfrac{\lambda}{\lambda+\mu}+\lambda t]e^{-\lambda t}, & t > 0. \end{cases} \qquad (5)$$

Hence we have

$$p_{\eta_1}(t) = \begin{cases} \dfrac{\lambda^2\mu}{(\lambda+\mu)^2}e^{\mu t}, & t<0, \\ \dfrac{\lambda^2\mu}{\lambda+\mu}\left[\dfrac{1}{\lambda+\mu}+t\right]e^{-\lambda t}, & t>0. \end{cases} \qquad (6)$$

and

$$\tilde{K}(\theta|z) = \frac{1-\varphi(\theta)}{\theta} + \varphi(\theta)\frac{\lambda^2\mu}{(\lambda+\mu)^2}e^{-\mu z}\int_{y=0}^{z} e^{\mu y}\tilde{K}(\theta|y)e^{\mu y}dy +$$
$$+\varphi(\theta)\frac{\lambda^2\mu}{(\lambda+\mu)^2}e^{\lambda z}\int_{y=z}^{a} e^{-\lambda y}\tilde{K}(\theta|y)dy -$$
$$-\varphi(\theta)\frac{\lambda^2\mu z}{\lambda+\mu}e^{-\lambda z}\int_{u=0}^{\phi} e^{-\lambda y}\tilde{K}(\theta|y)dy +$$
$$+\varphi(\theta)\frac{\lambda^2\mu}{\lambda+\mu}e^{\lambda z}\int_{y=z}^{a} e^{-\lambda y}y\tilde{K}(\theta|y)dy. \qquad (7)$$

We denote:

$$\int_0^{\infty}\mathrm{P}\{\tau > t|X(0)=z\}dt = E(\tau|X(0)=z)$$

$$\tilde{K}(\theta|z) = \int_0^{\infty} e^{\theta t}\mathrm{P}\{\tau > t|X(0)=z\}dt,$$

$$L(\theta|z) = 1 - \theta\tilde{K}(\theta|z). \qquad (8)$$

We can write, the equation (8) in the following form using (7):

$$L(\theta|z) = \varphi(\theta) - \varphi(\theta)\frac{\lambda^2\mu}{(\lambda+\mu)^2}e^{-\mu z}\int_{y=0}^{z} e^{\mu y}dy +$$
$$+\varphi(\theta)\frac{\lambda^2\mu}{(\lambda+\mu)^2}e^{-\mu z}\int_{y=0}^{z} e^{\mu y}L(\theta|y)e^{\mu y}dy -$$
$$-\varphi(\theta)\frac{\lambda^2\mu}{(\lambda+\mu)^2}e^{\lambda z}\int_{y=0}^{a} e^{-\lambda y}dy +$$
$$+\varphi(\theta)\frac{\lambda^2\mu}{(\lambda+\mu)^2}e^{\lambda z}\int_{y=z}^{a} e^{-\lambda y}L(\theta|y)dy + \varphi(\theta)\frac{\lambda^2\mu z}{\lambda+\mu}e^{\lambda z}\int_{y=z}^{a} e^{-\lambda y}dy -$$
$$+\varphi(\theta)\frac{\lambda^2\mu z}{\lambda+\mu}e^{\lambda z}\int_{y=z}^{a} e^{-\lambda y}L(\theta|y)dy - \varphi(\theta)\frac{\lambda^2\mu}{\lambda+\mu}e^{\lambda z}\int_{y=z}^{a} e^{-\lambda y}ydy +$$
$$+\varphi(\theta)\frac{\lambda^2\mu}{\lambda+\mu}e^{\lambda z}\int_{y=z}^{a} e^{-\lambda y}yL(\theta|y)dy. \qquad (9)$$

From (9) we can receive the differential equation:

$$L'''(\theta|z) - (2\lambda-\mu)L''(\theta|z) + \lambda(\lambda-2\mu)L'(\theta|z) +$$
$$+\lambda^2\mu[1-\varphi(\theta)]L(\theta|z) = 0. \qquad (10)$$

The characteristic equation of (10) will be in the following form

$$k^3(\theta) - (2\lambda-\mu)k^2(\theta) + \lambda(\lambda-2\mu)k(\theta) + \lambda^2\mu[1-\varphi(\theta)] = 0. \qquad (11)$$

Then the common solution of (10) will be

$$L(\theta|z) = \sum_{i=1}^{3} d_i(\theta)e^{k_i(\theta)}. \qquad (12)$$

From (9) we can find the initial conditions for differential equation (10) :

$$\begin{cases} L(\theta|a) = \varphi(\theta) - \varphi(\theta)\dfrac{\lambda^2\mu}{(\lambda+\mu)^2}e^{-\mu a}\displaystyle\int_{y=0}^{a} e^{\mu y}dy + \\ +\varphi(\theta)\dfrac{\lambda^2\mu}{(\lambda+\mu)^2}e^{-\mu a}\displaystyle\int_{y=0}^{a} e^{\mu y}L(\theta|y)dy, \\ L'(\theta|a) = \varphi(\theta)\dfrac{\lambda^2\mu^2}{(\lambda+\mu)^2}e^{-\mu a}\displaystyle\int_{y=0}^{a} e^{\mu y}dy - \\ -\varphi(\theta)\dfrac{\lambda^2\mu^2}{(\lambda+\mu)^2}e^{-\mu a}\displaystyle\int_{y=0}^{a} e^{\mu y}L(\theta|y)dy, \\ L''(\theta|a) = -\mu L'(\theta|a). \end{cases} \qquad (13)$$

From (13) we can receive the following system of the linear algebraic equations for $d_1(\theta), d_2(\theta)$ and $d_3(\theta)$.

$$\begin{cases} [[(\lambda+\mu)^2-(\mu+k_2(\theta))(\mu+k_3(\theta))]e^{k_1(\theta)a}+ \\ (\mu+k_2)(\mu+k_3)e^{\mu a}]d_1(\theta)+ \\ [[(\lambda+\mu)^2-(\mu+k_1(\theta))(\mu+k_3(\theta))]e^{k_2(\theta)a}+ \\ +(\mu+k_1)(\mu+k_3)e^{\mu a}]d_2(\theta)+ \\ [[(\lambda+\mu)^2-(\mu+k_1(\theta))(\mu+k_2(\theta))]e^{k_3(\theta)a}+ \\ +(\mu+k_1(\theta))(\mu+k_2(\theta))e^{\mu a}]d_3(\theta)= \\ =\varphi(\theta)(2\lambda\mu+\mu^2-\lambda^2e^{-\mu a}), \\ [[(\lambda+\mu)^2k_1(\theta)+\mu(\mu+k_2(\theta))(\mu+k_3(\theta))]e^{k_1(\theta)a}- \\ -\mu(\mu+k_2)(\mu+k_3)e^{\mu a}]d_1(\theta)+ \\ [[(\lambda+\mu)^2k_2(\theta)+\mu(\mu+k_1(\theta))(\mu+k_3(\theta))]e^{k_2(\theta)a}- \\ -\mu(\mu+k_1)(\mu+k_3)e^{\mu a}]d_2(\theta)+ \\ [[(\lambda+\mu)^2k_3(\theta)+\mu(\mu+k_1(\theta))(\mu+k_2(\theta))]e^{k_3(\theta)a}- \\ -\mu(\mu+k_1)(\mu+k_2)e^{\mu a}]d_3(\theta)= \\ =\lambda^2\mu\varphi(\theta)(1-e^{-\mu a}), \\ (\mu+k_1(\theta))k_1(\theta)e^{k_1(\theta)a}+(\mu+k_2(\theta))k_2(\theta)e^{k_2(\theta)a}+ \\ +(\mu+k_3(\theta))k_3(\theta)e^{k_3(\theta)a}=0. \end{cases} \qquad (14)$$

To find $d_i(\theta), i=\overline{1,3}$ we must find $d_i(0), i=\overline{1,3}$.

It is obvious, that

$$L(\theta)=\int_{z=0}^{a}L(\theta|z)dP\{\min(a,\eta_1^+)<z\}=$$

$$=\int_{z=0}^{a}L(\theta|z)d\left[1-P\{\min(a,\eta_1^+)>z\}\right]= \qquad (15)$$

$$=L(\theta|0)P\{\eta_1^+>a\}-\int_{z=0}^{a}L(\theta|z)d_zP\{\eta_1^+>z\}$$

$$L(\theta)=L(\theta|a)(1+\lambda a)e^{-\lambda a}+\lambda^2\int_{z=0}^{a}ze^{-\lambda z}L(\theta|z)dz.$$

For applications we find the expectation and variance of the distribution of the random variable $\tau(\omega)$. We know that

$$E\tau(\omega)=L'(0)$$

From (15) we find that

$$L'(0)=-\frac{\lambda\mu a}{\lambda-2\mu}\varphi'(0)e^{-\lambda a}-\frac{2\mu}{\lambda-2\mu}\varphi'(0)e^{-\lambda a}+\frac{2\mu}{\lambda-2\mu}\varphi'(0)+\frac{2\varphi'(0)}{f_1}\times$$
$$\times\Bigg\{\frac{1}{4(\lambda-2\mu)}\left[(\lambda-\mu+b)(-\mu^3(2\lambda+\mu+b)^2+4\lambda^4\mu)+2\lambda^5(2\lambda-\mu-b)\right]e^{\frac{2\lambda-3\mu+b}{2}a}-$$
$$-\frac{\lambda\mu(2\lambda-\mu+b)a}{8(\lambda-2\mu)}\left[\mu^2(2\lambda+\mu+b)^2-4\lambda^4\right]e^{\frac{2\lambda-3\mu+b}{2}a}+$$
$$+\frac{1}{4(\lambda-2\mu)}\left[(\lambda-\mu-b)(\mu^3(2\lambda+\mu-b)^2-4\lambda^4\mu)-2\lambda^5(2\lambda-\mu-b)\right]e^{\frac{2\lambda-3\mu-b}{2}a}+$$
$$+\frac{\lambda\mu(2\lambda-\mu-b)a}{8(\lambda-2\mu)}\left[\mu^2(2\lambda+\mu-b)^2-4\lambda^4\right]e^{\frac{2\lambda-3\mu-b}{2}a}+$$
$$+\left[\frac{\lambda^2\mu^3(\lambda+\mu+b)}{\lambda-2\mu}+\frac{2\lambda^3\mu^4(\lambda+\mu+b)a}{(\lambda-2\mu)(\mu-b)}\right]e^{-\frac{3\mu-b}{2}a}-$$
$$-\left[\frac{\lambda^2\mu^3(\lambda+\mu-b)}{\lambda-2\mu}+\frac{2\lambda^3\mu^4(\lambda+\mu-b)a}{(\lambda-2\mu)(\mu+b)}\right]e^{-\frac{3\mu+b}{2}a}-$$
$$-\frac{\lambda\mu^3(2\lambda+\mu)b}{\lambda-2\mu}e^{-\mu a}-\left[\frac{\lambda^3(\lambda^2-3\mu^2)b}{\lambda-2\mu}+\lambda^4\mu ab-\lambda^5\mu a^2b\right]e^{(\lambda-2\mu)a}\Bigg\}$$

We know that

$$D\tau(\omega)=L''(0)-[L'(0)]^2.$$

The following fact is proved at $\lambda<2\mu$:

$\lambda<2\mu$	$E\tau(\omega)$
$a\to 0$	$-\varphi'(0)>0$
$a\to\infty$	$\frac{2\mu}{\lambda-2\mu}\varphi'(0)>0$

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