Asymptotic Properties of the Branching Processes with State – Dependent Immigrations

Jakhongir Azimov
Institute of Mathematics, National University of Uzbekistan, Tashkent, Uzbekistan
jakhongir20@rambler.ru

***Abstract*— We consider the branching processes with state-dependent immigration and limit theorems for such processes. Assume that the intensity of the immigration decreases tending to 0, when the number of descendent increases.**

***Keywords*— branching process, immigration, stationary measure, slowly varying function,**

I. Introduction

We assume that:

a) $X = \{X_{in};\ i = 1,2,...,\ n = 0,1,...\}$

is a set of independent and identically distributed (i.i.d.) random variables with probability generating function (g.f.)

$$F(s) = \sum_{j=0}^{\infty} p_j s^j, \quad |s| \le 1,\ p_j \ge 0,\ \sum_{j=0}^{\infty} p_j = 1;$$

b) $Y = \{Y_n;\ n = 0,1,...\}$ is (independent of X) set of independent random variables with probability g.f.:

$$G_n(s) = \sum_{j=0}^{\infty} q_j(n) s^j, \quad |s| \le 1, \qquad q_j \ge 0,$$

$$\sum_{j=0}^{\infty} q_j(n) = 1, \qquad n = 0,1,2,\dots .$$

We define the branching random process $\{Z_n\}_{n=0}^{\infty}$ as follows:

$$Z_0 = 0, \quad Z_{n+1} = \sum_{i=1}^{Z_n} X_{in} + Y_n I_{\{Z_n=0\}}$$

where $\sum_{i=1}^{0} \cdot = 0$, and $I_{\{Z_n=0\}}$ – is indicator.

Suppose, that

$$F(s) = s + (1-s)^{1+\nu} L(1-s), \qquad (1)$$

where $0 < \nu \le 1$ and $L(s)$ is a slowly varying function (s.v.f.) as $s \to 0$.

It is known [1], that under the condition $0 < F(0) < 1$ there exists stationary measure for the Galton – Watson process, g.f. $U(s)$ of which is analytic in the circle $|s| < q$ (q-probability of degeneration) and in the case $U(F(0)) = 1$ the following Abel's functional equation holds:

$$U(F(s)) = 1 + U(s), \qquad |s| < q, \qquad U(1) = \infty.$$

Observe, that (1) implies (see [2])

$$U(s) = \frac{1+o(1)}{\nu(1-s)^{\nu} L(1-s)}, s \to 1 \qquad (2)$$

From the asymptotic relation (2) follows, that the inverse function of $U(1-x)$ has the following form

$$g(x) = \frac{N(x)}{x^{1/\nu}}, \qquad x > 0,$$

where $N(x)$ is a s.v.f. as $x \to \infty$ such that

$\nu N^{\nu}(x) L(x^{-1/\nu} N(x)) \to 1$.

II. Main Results

Denote

$$\alpha_n = EY_n = G_n'(1), \qquad \beta_n = DY_n + \alpha_n^2 - \alpha_n,$$

$$Q_1(n) = \alpha_n \sum_{k=0}^{n} (1 - F_k(0)),$$

$$Q_2(n) = (1 - F_n(0)) \sum_{k=0}^{n} \alpha_k.$$

where $F_0(s) = s, \quad F_{n+1}(s) = F\left(F_n(s)\right)$.

We suppose that

$$\sup_n \alpha_n < \infty, \qquad \sup_n \beta_n < \infty,$$

$$0 < \alpha_n \to 0, \qquad \beta_n \to 0, \qquad n \to \infty$$

Introduce the function

$$M(n) = \sum_{k=1}^{n} \frac{N(k)}{k^{1/\nu}}$$

We consider the case of $M(n) \to M < \infty$ as $n \to \infty$.

Theorem 1. Assume that

$$\alpha_n \sim \frac{l(n)}{n^r}, \ \beta_n = o\big(Q_1(n)\big), \quad n \to \infty,$$

where $r > 0$ and $l(n)$ is a s.v.f. as $n \to \infty$, if $r = 0$ then $l(n) = o(1)$, $n \to \infty$,

and

$$\theta_n = \frac{Q_1(n)}{Q_2(n)} \to \infty \text{ as } n \to \infty.$$

Then the following limits are finite

$$\lim_{n\to\infty} P\{Z_n = k \mid Z_n > 0\} = P_k^*, \quad \sum_{k=1}^{\infty} P_k^* = 1 \text{ and g.f.}$$

$$\varphi(s) = \sum_{k=1}^{\infty} P_k^* s^k = 1 - \frac{1}{M} \sum_{k=1}^{\infty} g(k + U(s)). \qquad (3)$$

Theorem 2. Let conditions of Theorem 1 hold and $\theta_n \to \theta, \quad 0 < \theta < \infty$ as $n \to \infty$.

Then the following limits

$$\lim_{n\to\infty} P\{Z_n = k \mid Z_n > 0\} = p_k, \quad \sum_{k=1}^{\infty} p_k = \frac{\theta}{1+\theta} < 1 \quad \text{exist,}$$

and $\sum_{k=1}^{\infty} p_k s^k = \dfrac{\theta}{1+\theta} \varphi(s)$.

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