On the Convergence of Payoffs

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***Abstract*— The problem of optimal stopping with incomplete data is reduced to the optimal stopping problem with complete data and the convergence of payoffs is proved when the small perturbation parameter of the observable process tends to zero.**

***Keywords*— partially observable process, gain function, payoff, optimal stopping.**

1. Let $(\Omega, \mathrm{F}, \mathrm{F}_t, P)$ be a complete probability space with filtration and suppose that on this space the system of stochastic differential equations is given

$$d\theta_t = a(t)\theta_t\,dt + b(t)dw_1(t), \quad \theta_0 = 0, \tag{1}$$

$$d\xi_t = a(t)\theta_t\,dt + \varepsilon\,dw_2(t), \quad \xi_0 = 0, \tag{2}$$

where $\varepsilon > 0$, the deterministic functions $a(t)$, $b(t)$ are continuous and measurable on $[0,T]$, w_1, w_2 are independent Wiener processes. It is assumed that θ_t is the nonobservable process and ξ_t is the observable process [1]. Consider a linear gain function

$$g(t,x) = f_0(t) + f_1(t)x, \quad x \in R, \tag{3}$$

where $f_0(t)$, $f_1(t)$ are measurable functions and introduce the payoffs

$$s^0 = \sup_{\tau \in \mathrm{M}^{\theta}} Eg(\tau, \theta_\tau), \qquad s^\varepsilon = \sup_{\tau \in \mathrm{M}^{\xi}} Eg(\tau, \theta_\tau), \tag{4}$$

where M^{θ}, M^{ξ} are the classes of stopping times with respect to the families of σ-algebras

$$\mathrm{F}_t^{\theta} = \sigma\{\theta_s, s \le t\}, \ \mathrm{F}_t^{\xi} = \sigma\{\xi_s, s \le t\}.$$

The problem of optimal stopping with incomplete data are reduce to the problem with complete data and the prove of convergence $s^\varepsilon \to s^0$ when $\varepsilon \to 0$ [2].

2. Let the following notations are introduced:

$$m_t = E\left(\theta_t \,|\, \mathrm{F}_t^{\xi}\right), \quad \gamma_t = E\left(\theta_t - m_t\right)^2 \tag{5}$$

and assume that the following conditions are satisfied

$$\int_0^T a^2(t)dt < \infty, \quad \int_0^T b^2(t)dt < \infty. \tag{6}$$

Lemma. *Let the partially observable process* (θ_t, ξ_t), $0 \le t \le T$, *is given by system* (1), (2) *and conditions* (6) *are satisfied. Then*

$$dm_t = a(t)m_t\,dt + \frac{a(t)\gamma_t}{\varepsilon^2}\left(d\xi_t - a(t)m_t\,dt\right), \tag{7}$$

$$\gamma_t' = 2a(t)\gamma_t - \frac{a^2(t)\gamma_t^2}{\varepsilon^2} + b^2(t), \tag{8}$$

where $m_0 = E(\theta_0 \,|\, \xi_0) = 0$, $\gamma_0 = E(\theta_0 - m_0)^2 = 0$.

Lemma is proved similarly to Theomem 10.1 in [1]

Theorem 1. *Let* $a(t)$, $b(t)$, $f_0(t)$, $f_1(t)$ *are bounded functions on* $[0,T]$. *Then:*

$$s^\varepsilon = \sup_{\tau \in \mathrm{M}^{\xi}} Eg(\tau, m_\tau), \tag{9}$$

$$s^\varepsilon = \sup_{\tau \in \mathrm{M}^{\theta}} Eg(\tau, \eta_\tau), \tag{10}$$

where the process η_t, $0 \le t \le T$, *is given by following equation*

$$d\eta_t = a(t)\eta_t\,dt + \frac{a(t)\gamma_t}{\varepsilon}dw_1(t). \tag{11}$$

Proof. 1. Let σ-algebra F_τ^{ξ} consists of the following events $A \in \mathrm{F}$ for which $A \cap \{\tau \le t\} \in \mathrm{F}_t^{\xi}$. Then

$$s^\varepsilon = \sup_{\tau \in \mathrm{M}^{\xi}} Eg(\tau, \theta_\tau) = \sup_{\tau \in \mathrm{M}^{\xi}} E\left\{Eg(\tau, \theta_\tau) / \mathrm{F}_\tau^{\xi}\right\} = \\ = \sup_{\tau \in \mathrm{M}^{\xi}} E\left\{f_0(\tau) + f_1(\tau)E\left(\theta_\tau \,|\, \mathrm{F}_\tau^{\xi}\right)\right\}.$$

According Lemma 1.9 in [1] on the set $\{\tau = t\}$ we have $E\left(\theta_\tau \,|\, \mathrm{F}_\tau^{\xi}\right) = E\left(\theta_t \,|\, \mathrm{F}_t^{\xi}\right)$ by which we obtain (9).

2. According to Theorem 7.12 in [1] we have

$$d\xi_t = a(t)m_t\,dt + \varepsilon\,d\bar{w}(t), \tag{12}$$

where $\bar{w}(t)$ so-called innovative Wiener process for which $\mathrm{F}_t^{\bar{w}} = \mathrm{F}_t^{\xi}$. From (7), (12) we have

$$dm_t = a(t) m_t \, dt + \frac{a(t)\gamma_t}{\varepsilon} d\bar{w}(t) \tag{13}$$

and from (11), (13) we have $\mathrm{F}_t^{\eta} = \mathrm{F}_t^{\theta}$ and $\mathrm{M}^{\eta} = \mathrm{M}^{\theta}$ by which we obtain (10).

Theorem 2. Let $\rho(t)$ be a continuous increasing function, majorizing the function $|b(t)|\Phi_t^{-2}/a(t)$, where

$$\Phi(t) = \exp\left\{\int_0^t a(s)\,ds\right\}.$$

Then

$$\gamma_t \le \varepsilon \Phi_t^2 \rho(t), \quad t \le T. \tag{14}$$

Proof. Let us make a transformation

$$\gamma_t = \varepsilon \Phi_t^2 u(t).$$

Easy to see that

$$u'(t) = \frac{1}{\varepsilon} a^2(t) \Phi_t^2 \left[\frac{a^2(t) b^2(t)}{\Phi_t^4} - u^2(t)\right], \; u(0) = 0. \tag{15}$$

We show that $u(t) \le \rho(t)$. Consider the opposite. Suppose there exist t_0 and t_1, $t_0 < t_1$ such that $u(t_0) = \rho(t_0)$, $u(t) > \rho(t)$, when $t_0 < t < t_1$. From (15) we get

$$u'(t) \le \frac{1}{\varepsilon} a^2(t) \Phi_t^2 \left[\rho^2(t) - u^2(t)\right] < 0, \quad t \in (t_0, t_1].$$

Hence $u(t) < u(t_0) = \rho(t_0) \le \rho(t)$, i.e. $u(t) < \rho(t)$, which is a contradiction.

Theorem 3. *Let partially observable process* (θ_t, ξ_t), $0 \le t \le T$, *is given by system* (1), (2) *and conditions* (6) *are satisfied. Then*

$$\lim_{s \to 0} s^{\varepsilon} = \lim_{\varepsilon \to 0} \sup_{\tau \in \mathrm{M}^{\xi}} Eg(\tau, \theta_\tau) = \sup_{\tau \in \mathrm{M}^{\theta}} Eg(\tau, \theta_\tau) = s^0.$$

Directly proof we get by Theorems 1 and 2.

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