On an Application of Hybrid Method to the Solution of Volterra Nonlinear Integro-Differential Equation

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Abstract— **While constructing mathematical model of some phenomena of nature we meet the solution of initial value problem for Volterra integro-differential equation. Many papers have been devoted to the numerical solution of this problem. In these works, at first the integral is replaced by the integral sum then one of the well investigated numerical methods from the class of numerical methods for solving ordinary differential equations is applied to the obtained one. Notice that in the methods suggested in these papers, mainly a variable boundary integral sum assisting to increase the volume of calculations while parsing from one point to another one is used. Therefore, we try to construct a method that preserves constant amount of calculations at each point.**

Keywords— Volterra integro-differential equations; hybrid method; multistep methods; degree; stable

I. Introduction

As is known, there exist mainly two classes of numerical methods for solving ordinary differential equations called one-step and multistep methods. Each of these methods has its advantage and shortage. Recently the scientists have constructed the methods possessing the best properties of one-step and multi-step methods. In scientific references these methods are called hybrid methods. Taking into account advantage of these methods, here we suggest application of hybrid methods to numerical solution of Volterra nonlinear integro-differential equation.

Consider the following problem

$$y' = f(x,y) + \int_{x_0}^{x} K(x,s,y(s))ds, \qquad y(x_0) = y_0 \quad (x \in [x, X]). \tag{1}$$

Assume that problem (1) has a unique solution determined on the interval $[x_0, X]$ that by means of the constant step $h > 0$ is divided into N equal parts and the mesh points are defined in the form $x_i = x_0 + ih \quad (i = 0,1,2,...)$. Denote by y_m the approximate value of the solution of problem (1) by $y(x_m)$ its exact value at the point $x_m (m = 0,1,2,...)$. The numerical solution of problem (1) is reduced to the solution of the following problem:

$$y' = f(x,y) + \nu(x), \quad y(x_0) = y_0, \quad x_0 \le x \le X,$$

$$\nu(x) = \int_{x_0}^{x} K(x,s,y(s))ds. \tag{2}$$

Many papers of different authors were devoted to numerical solution of initial value problem for Volterra integro-differential equations. In these papers, the method of quadratures is mainly applied to the solution of problem (1) (see e.g. [1]-[5]). It is known that in using the method of quadratures the amount of calculations increases while passing from one point to another one. Different authors constructed the methods for removing this shortcoming (see [4]). The method suggested in this paper differs from other by the fact that here a hybrid method is applied to the numerical solution of problem (1).

II. Construction of a Hybrid Method

Consider an integral participating in problem (1) that is a variable boundary integral and as the result of integration is a function dependent on a variable x. Consequently the denotation of (2) is verified. Assume that the function $\nu(x)$ is known and to the numerical solution of problem (1) apply the known multistep method with constant coefficients. As the result we get (see [5])

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i} + h \sum_{i=0}^{k} \beta_i \nu_{n+i}, \tag{3}$$

where $\alpha_i, \beta_i \ (i = 0,1,...,k)$ are some real numbers, moreover $\alpha_k \ne 0$, but

$$f_m = f(x_m, y_m), \quad \nu_m = \nu(x_m) \ (m = 0,1,2,...).$$

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The quantity k is an order of the difference equation in so far as some authors call the multistep methods with constant coefficients the finite difference method (see [5]).

Notice that in almost all the papers devoted to the solution of problem (1), a method of type (3) is obtained by means of multistep methods. But they differ from other by the scheme of calculation of the values of the function $v(x)$. Here, to this end we use the following method:

$$\sum_{i=0}^{k}\alpha'_i v_{n+i} = h\sum_{j=0}^{k}\sum_{i=0}^{k}\gamma_i^{(j)}K(x_{n+j},x_{n+i},y_{n+i}) \quad (4)$$

that was investigated in [4]. However, the method used here belongs to the class of hybrid methods and therefore differs from method (4).

Consider the case when $v(x) \equiv 0$, then problem (1) turns into a single initial value problem for ordinary differential equations. In [7] a hybrid method is constructed for numerical solution of such a problem. This method is A-stable uses the calculation of the function $f(x,y)$ at two point and has higher accuracy among the methods with similar properties. This method has the following form

$$y_{n+1} = y_n + h(3f_{n+\frac{1}{3}} + f_{n+1})/4, \quad (5)$$

where

$$f_{m+\alpha} = f(x_m + \alpha h, y_{m+\alpha}) \ (m = 0,1,2,...), \text{ and } \alpha \in (0,1).$$

A method for finding the values of the function $v(x)$ by means of relations (5) may be constructed in different variants one of which has the following form:

$$\begin{aligned} v_{n+1} = v_n + h(2K(x_{n+1/3}, x_{n+1/3}) + \\ + K(x_{n+1}, x_{n+1/3}), y_{n+1/3}) + \\ + K(x_{n+1}, x_{n+1}, y_{n+1}))/4. \end{aligned} \quad (6)$$

It is now from the problem that $v_0 = v(x) = 0$. Therefore in method (6) the quantity v_n may be assumed to be known. For using method (6), at first we can find $y_{n+1/3}$, and then calculate the quantity v_{n+1} that depends on the values of y_{n+1}. The accuracy of method (6) coincides with the accuracy of method (5) that is determined by the values of degree of method (5). Usually, degree of method (3) is determined as follows. They say that the integer parameter p is a degree of method (3) if it holds:

$$\begin{aligned} \sum_{i=0}^{k}(\alpha_i y(x+ih) - h\beta_i y'(x+ih)) = \\ = O(h^{p+1}), h \to 0, \end{aligned} \quad (7)$$

where $x = x_0 + nh$ is a fixed point.

It is easy to show that method (6) has degree $p = 3$. Consequently, the quantity $y_{n+1/3}$ should be determined so that the accuracy of method (6) be presented. This means that the method applied to the calculation of $y_{n+1/3}$ should have degree $p \ge 2$, in so far as the obtained value is taken into account in method (6) by means of the predictor-corrector method (see [7]). A numerical method with degree $p = 2$ using the calculation of the function $f(x,y)$ at two points is the trapezoid method that in the present case has the following form

$$\begin{aligned} y_{n+1/3} = y_n + h(f_n + f_{n+1/3})/6 \\ + h(v_n + v_{n+1/3})/6. \end{aligned} \quad (8)$$

In this method, the quantities $f_{n+1/3}$ and $v_{n+1/3}$ whose calculations depend on the values of $y_{n+1/3}$, are unknown. However accuracy of the value of the quantity $y_{n+1/3}$ doesn't change if the unknowns participating on the right hand side of method (8) are calculated by the methods possessing degree $p \ge 1$. Subject to what has been said, consider the following Euler method:

$$\bar{y}_{n+1/3} = y_n + h(f_n + v_n)/3. \quad (9)$$

Revise the obtained value by means of the following methods

$$\begin{aligned} \bar{v}_{n+1/3} = v_n + h(K(x_n, x_n, y_n) + \\ + K(x_n + h/3, x_n + h/3, \bar{y}_{n+1/3}))/6, \end{aligned} \quad (10)$$

$$\begin{aligned} \hat{y}_{n+1/3} = y_n + h(f_n + \bar{f}_{n+1/3})/6 + \\ + h(v_n + \hat{v}_{n+1/3})/6. \end{aligned} \quad (11)$$

where

$$f_{m+\alpha} = f(x_{m+\alpha}, \bar{y}_{m+\alpha}), \ m = 0,1,2,...; 0 \le \alpha \le 1).$$

Calculation of the value of the quantity v_{n+1} by the method (6) depends on the value of the quantity y_{n+1}. To this end we use the following predictor-corrector scheme:

$$\hat{v}_{n+1} = v_n + h(K(x_n, x_n, y_n) + + K(x_{n+1}, x_{n+1}, \bar{y}_{n+1}))/2, \quad (12)$$

$$\hat{y}_{n+1} = y_n + h(f_n + \bar{f}_{n+1})/2 + + h(v_n + \hat{v}_{n+1})/2, \quad (13)$$

where $\bar{y}_{n+1} = y_n + h(f_n + v_n)$.

Allowing for the schemes suggested above, rewrite method (6) in the following form:

$$v_{n+1} = v_n + h(2K(x_{n+1}, x_{n+1/3}, \hat{y}_{n+1/3}) + K(x_{n+1/3}, x_{n+1/3}, \hat{y}_{n+1/3}) + + K(x_{n+1}, x_{n+1}, \hat{y}_{n+1}))/4. \quad (14)$$

For calculating the values of the quantity y_{n+1} we suggest the following method:

$$y_{n+1} = y_n + h(3\hat{f}_{n+1/3} + \hat{f}_{n+1})/4 + + h(3\hat{v}_{n+1/3} + v_{n+1})/4. \quad (15)$$

We have noticed above that method (5) has degree $p = 3$ the method of trapezoid has degree $p = 2$, Euler's explicit method has degree $p = 1$. Thus we obtain that accuracy of approximate values of quantities y_{n+1} and v_{n+1} gradually increase by means of different degree methods. Usually, accuracy of the scheme suggested above is determined by means of degree of predictor-corrector methods that in the general form was investigated by different authors (see [9], [10]). Notice that in spite of the fact that during using the predictor-corrector method the accuracy of the method is preserved, however there many happen that some of its properties such as A-stability, can't be preserved. But application of the predictor-corrector method for realization of implicit methods usually allows extending their stability areas of such methods as the Simpson method, the forward jumping methods and etc. (see [11], [12]).

The scheme described above for defining the quantity y_{n+1} on the face of it of a bulky form and therefore we try to simplify it. To this end to definition of the quantity y_{n+1} we apply the following method:

$$y_{n+1} = y_n + + h(3f(x_n + h/3, (4y_n + 5y_{n+1})/9 - - 2hf_{n+1}/9) + f_{n+1})/4 + + h(3v(x_n + h/3) + v(x_n + h))/4. \quad (16)$$

This method differs from method (5) by the fact that here we don't use the $y_{n+1/3}$ values of function $y(x)$ of the solution of problem (1) at the point $x_n + h/3$. Now, consider determination of the quantities $v_{n+1/3}$ and v_{n+1}. At first consider determination of the quantities v_{n+1} and to this end use the following method:

$$v_{n+1} = v_n + h(2K(x_{n+1/3}, x_{n+1/3}, (4y_n + 5y_{n+1})/9 - - 2hf_{n+1}/9 - 2hv_{n+1}/9) + + K(x_{n+1}, x_{n+1/3}, (4y_n + 5y_{n+1})/9 - - 2hf_{n+1}/9 - 2hv_{n+1}/9) + + K(x_{n+1}, x_{n+1}, y_{n+1}))/4. \quad (17)$$

To determination of the quantities $v_{n+1/3}$ we apply the following method:

$$v_{n+1/3} = v_n + h(K(x_n, x_n, y_n) + + K(x_{n+1/3}, x_{n+1/3}, (4y_n + 5y_{n+1})/9 - - 2hf_{n+1}/9 - 2hv_{n+1}/9))/6. \quad (18)$$

Thus for determination of the quantities $v_{n+1/3}$ and v_{n+1} we obtain nonlinear algebraic equation, since the functions $f(x, y)$ and $K(x, s, y)$, generally speaking, are nonlinear.

For solving this equation we use iteration method. However for attaining the required accuracy at each step we encounter multiple calculations of the functions $f(x, y)$ and $K(x, s, y)$. Therefore, for determination of the quantities $v_{n+1/3}$ and v_{n+1} we suggest using the predictor-corrector scheme:

$$v_{n+1} = v_n + h(2K(x_{n+1/3}, x_{n+1/3}, (4y_n + 5\hat{y}_{n+1})/9 - - 2h(\hat{f}_{n+1} + \hat{v}_{n+1}/9) + + K(x_{n+1}, x_{n+1/3}, (4y_n + 5\hat{y}_{n+1}))/9 - - 2h(\hat{f}_{n+1} + \hat{v}_{n+1})/9) + + K(x_{n+1}, x_{n+1}, \hat{y}_{n+1}))/4, \quad (19)$$

$$v_{n+1/3} = v_n + h(K(x_n, x_n, y_n) + + K(x_{n+1/3}, x_{n+1/3}, (4y_n + 5y_{n+1})/9 - - 2h(\hat{f}_{n+1} + \hat{v}_{n+1})/9))/6, \quad (20)$$

where the quantities $\hat{v}_{n+1}$ and $\hat{y}_{n+1}$ are calculated by methods (12) and (13), respectively.

After finding the quantities indicated above, for determination of the quantities y_{n+1} we suggest the following method:

$$y_{n+1} = y_n + h(3f(x_{n+1/3}, (4y_n + 5\hat{y}_{n+1})/9 - 2h\hat{f}_{n+1}/9) + \hat{f}_{n+1})/4 + h(3v_{n+1/3} + v_{n+1})/4. \quad (21)$$

Now construct algorithms for using the above suggested methods.

To approximate the solution of problem (1)

$$y' = f(x,y) + \int_{x_0}^{x} K(x,s.y(s))ds,$$
$$x_0 \le x \le X,\ y(x_0) = y_0,$$

at $(N+1)$ equally spaced numbers in the interval $[x_0, X]$:

A. Algorithm I

INPUT endpoints x_0, X; positive integer N; initial condition y_0, function $f(x,y)$ and $K(x,s,y)$.

OUTPUT approximation y_i to $y(x_i)$.

STEP 1 Set $h = (x - x_0)/N$.

STEP 2 For $n = 0,1,2,...,N-3$ do STEP 3-8.

STEP 3 Set $x_n = x_0 + nh$.

STEP 4 Set $\bar{y}_{n+1/3}, v_{n+1/3}, \hat{y}_{n+1/3}$ (Compute by formulas (9), (10) and (11)).

STEP 5 Set $\bar{y}_{n+1} = y_n + h(f_n + v_n)$.

STEP 6 Set $\hat{v}_{n+1}, \hat{y}_{n+1}$ (Compute by formulas (12) and (13)).

STEP 7 Set v_{n+1}, y_{n+1} (Compute by formulas (14) and (15)).STEP 8 OUTPUT $(x_{n+1}; y_{n+1})$.

STEP 9 STOP.

Algorithm II

STEP 1 Set $h = (x - x_0)/N$.

STEP 2 For $n = 0,1,2,...,N-1$ do STEP 3-7.

STEP 3 Set $x_n = x_0 + nh$.

STEP 4 Set $\bar{y}_{n+1} = y_n + h(f_n + v_n))$.

STEP 5 Set $v_{n+1}, v_{n+1/3}$ (Compute by formulas (19) and (20)).

STEP 6 Set y_{n+1} (Compute by formula (21)).

STEP 7 OUTPUT $(x_{n+1}; y_{n+1})$.

STEP 8 STOP.

III. CONCLUSION

In these report we constructed some concrete method and algorithms for solving initial value problem for integro-differential equations. We can generalize these methods in the next form:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \beta_i f_{n+i+\eta_i} + h\sum_{i=0}^{k} \beta_i\, v_{n+i+\eta_i},$$

$$\sum_{i=0}^{k} \alpha_i v_{n+i+\xi_i} = $$
$$= h\sum_{j=0}^{k}\sum_{i=0}^{k} \gamma_{i,j}^{(j)} K(x_{n+j+l_i}, x_{n+i+m_i}, y_{n+i+m_i}).$$

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