On One Investigation of Numerical Solution of Integro-Differential Equations of Volterra Type

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***Abstract*— Starting with Volterra's paper, the scientists investigate the solutions of scientific-technical problems by means of integro-differential equations. For finding the approximate solutions of such equation, the quadratures method suggested by Volterra were mainly used. Taking into account some advantages of hybrid methods Makroglou applied them to numerical solution of integro-differential equation. Developing the indicated idea Makroglou constructed a new hybrid method with higher accuracy and this method was applied to the numerical solution of Volterra type integro-differential equation.**

***Keywords*— integro-differential equation; hybrid method; differential equation; degree and convergence**

I. INTRODUCTION

As you know, beginning from 1908, Volterra employed all effort for investigating the residual operation while creating theory of distortions. Volterra also researched some problems of ecology as well, considered population of animals and while constructing mathematical models of these and other problems, he obtained integro-differential equations. Volterra studied in detail the problem on elastic equilibrium of the isotropic globe involving memory, since elastic properties of the earth's crust preserve residual effects responding to large time intervals (see e.g. [1, p.50-55], [2, pp. 50-55]). Hence it is seen that the solution of integro-differential equations forms a basic while investigating many applied and scientific problems.

Consider the following integro-differential equation:

$$y' = f(x, y) + \int_{x_0}^{x} K(x, s, y(s))ds, \ x_0 \le s \le x \le X. \tag{1}$$

Note that equation (1) is said to be a Volterra type nonlinear integro-differential equation. Assume that on the segment $[x_0, X]$ equation (1) has a unique continuous solution satisfying the following condition:

$$y(x_0) = y_0. \tag{2}$$

In this paper we investigate numerical solution of problem (1) - (2). Therefore, by means of the constant step h we partitions in the form the segment $[x_0, X]$ into N equal parts and define the mesh points in the form:

$$x_i = x_0 + ih \quad (i = 0,1,2,..., N).$$

Denote the y_i approximate value, and by $y(x_i)$ the exact value of the solution of problem (1) - (2) at the points $x_i \ (i = 0,1,..., N)$.

It is known that one of the popular methods for solving the problem (1) - (2) is the quadrature method, in which the integral participating in equation (1) is replaced the integral sum, as a result we get:

$$y'(x_n) = f(x_n, y(x_n)) + \sum_{i=0}^{n} a_i K(x_n, x_i, y(x_i)) + R_n, \tag{3}$$

Here the coefficients of the quadrature formula $a_i \ (i = 0,1,2,..., n)$ are some real numbers, R_n is a residual term of the quadrature method. In equality (3) we reject the residual term R_n, add initial conditions (2) and get an ordinary differential problem that was thoroughly researched by many authors (see e.g. [3-8]). It is easy to notice that when the value of the quantity n increases, the sum participating in formula (3) is calculated again, consequently the volume of calculation increases according to increase of the values of the quantity n. For removing the mentioned shortcomings of the quadrature method, the specialist suggested different methods (see e.g. [9] - [13]). In [14] the following method is suggested:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \beta_i f_{n+i} + h\sum_{i=0}^{k} \beta_i \vartheta_{n+i}, \tag{4}$$

$$\sum_{i=0}^{k} \hat{\alpha}_i \vartheta_{n+i} = h\sum_{j=0}^{k}\sum_{i=0}^{k} \gamma_i^{(j)} K(x_{n+j}, x_{n+i}, y_{n+i}), \tag{5}$$

where $\alpha_i, \beta_i, \hat{\alpha}_i, \gamma_i^{(j)} \ (i, j = 0,1,2,...k)$ are some real numbers, moreover $\alpha_k \ne 0, \ \hat{\alpha}_k \ne 0$, and the function $\vartheta(x)$ is determined in the following form:

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$$\vartheta(x) = \int_{x_0}^{x} K(x, s, y(s))ds\,. \tag{6}$$

In order to simplify the indicated procedure and construct the higher accuracy methods, we suggest applying hybrid methods to solving problem (1) - (2). As it was noted, the numerical solution of (1) - (2) by means of hybrid methods was investigated by Makroglou (see e.g. [15]), where the following method was applied to the solution of problem (1) - (2):

$$\sum_{i=0}^{k} \alpha_i z_{n+i} = h\sum_{i=0}^{k} \beta_i z'_{n+i} + h\beta_\nu z'_{n+\nu}. \tag{7}$$

In this paper, for numerical solution of problem (1) - (2), we offer to apply the following method:

$$\sum_{i=0}^{k} \alpha_i z_{n+i} = h\sum_{i=0}^{k} \beta_i z'_{n+i+\nu_i} \quad (|\nu_i| < 1; i = 0,1,...,k) \tag{8}$$

Note that for $\nu_i = 0 \quad (i = 0,1,...,k)$ from the finite-difference method one can get the k-step method with constant coefficients that were thoroughly researched by the famous scientists as G.Dalquist, J.Butcher, P.Henriki, N.Bakhvalov and others. However, as Dalquist proved the degree of stable k-step method with constant coefficients satisfies the condition $p \le k+2$, but the degree of the stable method of type (8) satisfies the condition $p \le 2k+2$ (here p denotes the method's degree, i.e. the accuracy degree). Thus we see that the method (8) represents both scientific and practical interest. The last years many authors try to reduce the solution of integro-differential equations to systems composed of differential integral differential and integral equations. Therefore, we consider the relation between the indicated method.

II. Comparison of Some Methods for Solving Integro-Diggerential Equation

It is easy to define that the solution of equation (1) may be replaced by the solution of the following system of equations:

$$y' = f(x, y) + \vartheta(x), \tag{9}$$

$$\vartheta(x) = \int_{x_0}^{x} K(x, s, y(s))ds. \tag{10}$$

If we find a solution the solution of equation (10) and take it into account (9), then we get obtain an ordinary differential equation and apply method (8) to its solution. Consequently, solving equation (10) with accuracy order $2k+2$, we can construct a method with accuracy order $2k+2$ for solving problem (1) - (2).

Consider the case when the kernel of the integral of the functions $K(x, s, y)$ may be represented in the form $K(x, s, y) = a(x)b(s, y)$. Then we can replace system of problems

$$y' = f(x, y) + a(x)\vartheta(x), \quad y(x_0) = y_0, \tag{11}$$

$$\vartheta' = b(x, y), \quad \vartheta(x_0) = 0\,, \; x \in [x_0, X]\,. \tag{12}$$

Applying method (8) to solving problems (11) and (12), one can find the approximate value of the solution of problem (1) - (2) with accuracy $p = 2k+2$.

However, the solutions of problem (1) - (2) may be reduced to the solution of the system of integral equations of the following form:

$$y(x) = y(x_0) + \int_{x_0}^{x} f(s, y(s))ds + \int_{x_0}^{x} \vartheta(s)ds, \tag{13}$$

$$\vartheta(x) = \int_{x_0}^{x} K(x, s, y(s))ds, \; x \in [x_0, X] \tag{14}$$

The obtained system consists of only of integral equations.

Thus, we see that using one of the above-described schemes, we can solve problem (1) - (2). Here we investigate the solution of problem (1) - (2) by means of the system of equations (9) - (10) or by means of the system of equations (13) - (14). Obviously, therewith the accuracy of the constructed method for solving problem (1) - (2) mainly will depend on the accuracy of the method applied to the solution of integral equations. Therefore, consider the application of method (8) to the numerical solution of Volterra type integral equation. To this end we impose some restrictions on the coefficients of method (8). We can say that method (8) converges then the fulfillment of the following conditions is a natural restriction an its coefficients.

A. The values of the quantities $\alpha_i, \beta_i, \nu_i \; (i = 0,1,2,...,k)$ are real numbers, moreover $\alpha_k \ne 0$.

B. The characteristical polynomials

$$\rho(\lambda) \equiv \sum_{i=0}^{k} \alpha_i \lambda^i; \quad \sigma(\lambda) \equiv \sum_{i=0}^{k} \beta_i \lambda^{i+\nu_i}$$

Have no common multipliers different from a constant.

C. It holds: $\vartheta(1) \ne 0$ and $p \ge 1$.

These conditions may be proved by a scheme similar to one in the papers [3].

It is known that the accuracy of each method is determined by the values of its coefficients. Before determining the values of the quantities $\alpha_i, \beta_i, \nu_i \; (i = 0,1,2,...,k)$, we consider the definition of the degree of the method.

A is known the methods accuracy usually is defined by means of the notion of order of methods, however here the method's accuracy is define by means of the notion of the degree of the method since the order of method (8) is quantity k. This is related with the fact oftenly relation (8) is assumed to be a difference equation, and in this case k is the order of a difference equation. However, taking into account that $\alpha_k \neq 0$, then from equality (8) we can define the quantity y_{n+k}, but in this case relation (8) is oftenly called a difference method. We also will use this name in some.

Definition. Assume that the function $z(x)$ is sufficiently smooth. It is said that the integer quantity p is the degree of method (8), if it holds the following one:

$$\sum_{i=0}^{k} \alpha_i z(x+ih) - h\sum_{i=0}^{k} \beta_i z'(x+(i+\nu_i)h) = \\ = O(h^{p+1}),\ h \to 0 \tag{15}$$

here $x = x_0 + nh$ is a fixed point.

For determining the coefficients of method (8), different methods may be used, but here we use the method of undetermined coefficients.

Now consider the selection of the values of the quantities α_i, β_i, ν_i $(i = 0,1,2,...,k)$). To this end, usually the method of undetermined coefficients is used based on the Taylor formula and that at this case has the following form

$$y(x+ih) = y(x) + ihy'(x) + \frac{(ih)^2}{2!} y''(x) + ... + \\ + \frac{(ih)^p}{p!} y^{(p)}(x) + O(h^{p+1}), \tag{16}$$

$$y'(x+l_i h) = y'(x) + l_i h y''(x) + \frac{(l_i h)^2}{2!} y'''(x) + ... + \\ + \frac{(l_i h)^{p-1}}{(p-1)!} y^{(p)}(x) + O(h^p) , \tag{17}$$

where $x = x_0 + nh$ - is fixed point, but $l_i = i + \nu_i$ $(i = 0,1,2,...,k)$

For determining the values of the parameters α_i, β_i, l_i $(i = 0,1,2,...,k,\ l_i = i + \nu_i)$, take into account equalities (16) and (17) in asymptotic equality (15)? then have:

$$\sum_{i=0}^{k} \alpha_i = 0 ;$$

$$\sum_{i=0}^{k} \frac{i^\nu}{\nu!} \alpha_i = \sum_{i=0}^{k} \frac{l_i^{\nu-1}}{(\nu-1)!} \beta_i \quad (\nu = 1,2,..., p,\ 0! = 1) . \tag{18}$$

Thus, for determining the values of the α_i, β_i, l_i $(i = 0,1,2,...,k)$ we get a homogeneous system of nonlinear algebraic equation where the amount of unknowns equals $3k+3$, but the amount of equation $p+1$. Obviously system (18) always has a trivial solution. However the trivial solution of system (18) is not of interest. Therefore we consider the case when system (18) to have a non-zero solution. It is known, that the system (18) has the non zero solution if the condition

$$p < 3k + 2 \tag{19}$$

holds.

Hence it follows that $p_{\max} = 3k + 1$.

Construct a concrete method for $k = 2$. Then from system (18) we have:

$$\beta_2 + \beta_1 + \beta_0 = 2\alpha_2 + \alpha_1 ,$$
$$l_2\gamma_2 + l_1\beta_1 + l_0\beta_0 = \frac{1}{2}(2^2\alpha_2 + \alpha_1) ,$$
$$l_2^2\beta_2 + l_1^2\beta_1 + l_0^2\beta_0 = \frac{1}{3}(2^3\alpha_2 + \alpha_1) ,$$
$$l_2^3\beta_2 + l_1^3\beta_1 + l_0^3\beta_0 = \frac{1}{4}(2^4\alpha_2 + \alpha_1) , \tag{19}$$
$$l_2^4\beta_2 + l_1^4\beta_1 + l_0^4\beta_0 = \frac{1}{5}(2^5\alpha_2 + \alpha_1) ,$$
$$l_2^5\beta_2 + l_1^5\beta_1 + l_0^5\beta_0 = \frac{1}{6}(2^6\alpha_2 + \alpha_1) .$$

Selecting α_2, α_1 and α_0 we can construct different methods with the degree $p = 6$. For $\alpha_2 = 1$, $\alpha_1 = \alpha_0 = -1/2$ solving system (20) we get a complex solution, but for $\alpha_2 = 1$, $\alpha_1 = -1$ and $a_0 = 0$ the solution of system (20) has the following form:

$$\beta_2 = 5/18 ,\ \beta_1 = 8/18 ,\ \beta_0 = 5/18,$$
$$l_2 = 3/2 + \sqrt{15}/10,\ l_1 = 3/2,\ l_0 = 3/2 - \sqrt{15}/10 . \tag{20}$$

Taking into account these values in formula (8) we get the following method:

$$y_{n+2} = y_{n+1} + h(5y'_{n+3/2+\sqrt{15}/10} + \\ +10y'_{n+3/2} + 5y'_{n+3/2-\sqrt{15}/10})/18 . \tag{21}$$

This method is stable and has the degree $p = 6$. Construct an algorithm for using method (19). Assume that y_1 is known, and consider the calculation of y_{n+2} $(n = 0,1,2,...)$. For applying method (19) the values of the quantities $y_{n\pm\alpha}$ $(\alpha =)$, should be known and they are determined by means of the following formula:

$$y_{n+\alpha} = y_n + 2hy'_n + \alpha^2 h \times \\ \times((h^2 - 12\alpha + 6)y'_{n+3/2} - (3\alpha^2 - 48\alpha + 27)y'_{n+1} + \\ + (3\alpha^2 - 60\alpha + 54)y'_{n+1/2} - (\alpha^2 - 24\alpha + 33)y'_n)/18, \tag{22}$$

where $\alpha = 3/2 \pm \sqrt{15}/10,$

Thus, we see that for using method (21), we must construct formulae for calculating the values of $y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}$ with accuracy $O(h^5)$. If $y_{\frac{1}{2}}$ are know, then for $n>1$ it is enough to calculate the values of y_{n+1} and $y_{n+\frac{3}{2}}$. Therefore, at first consider the definition of the value of the quantity y_{n+1} and to this end we use the following sequential formulae:

1. $\breve{y}_{n+1} = y_n + hy'_n,$
2. $\bar{y}_{n+1} = y_n + h(y'_n + \breve{y}'_{n+1})/2,$
3. $\hat{y}_{n+1} = y_n + h(y'_n + 4y'_{n+\frac{1}{2}} + \bar{y}'_{n+1})/6,$
4. $y_{n+1} = y_n + h(y'_n + 4y'_{n+\frac{1}{2}} + \hat{y}'_{n+1})/6,$
5. $\hat{y}_{n+\frac{3}{2}} = y_{n+\frac{1}{2}} + h(7y'_{n+1} - 2y'_{n+\frac{1}{2}} + y'_n)/6,$
6. $y_{n+\frac{3}{2}} = y_{n+1} + h(9y'_{n+\frac{3}{2}} + 19y'_{n+1} - 5y'_{n+\frac{1}{2}} + y'_n)/48$

Now calculate the values of $y_{\frac{1}{2}}$.

1. $\bar{y}_{\frac{1}{4}} = y_0 + hy'_0/4,$
2. $\hat{y}_{\frac{1}{4}} = y_0 + h(y'_0 + \bar{y}'_{\frac{1}{4}})/8,$
3. $\bar{y}_{\frac{1}{2}} = y_0 + hy'_0/2,$
4. $\breve{y}_{\frac{1}{2}} = y_0 + h(y'_0 + \bar{y}'_{\frac{1}{2}})/4,$
5. $y_{\frac{1}{4}} = y_0 + h(5y'_0 + 8\hat{y}'_{\frac{1}{4}} - \breve{y}'_{\frac{1}{2}})/48,$
6. $\hat{y}_{\frac{1}{2}} = y_0 + h(y'_0 + 4y'_{\frac{1}{4}} + \breve{y}'_{\frac{1}{2}})/12,$
7. $y_{\frac{1}{2}} = y_0 + h(y'_0 + 4y'_{\frac{1}{4}} + \hat{y}'_{\frac{1}{2}})/12.$

If in method (21) we assume $n := n-1$, we can get a one step method of the following form:

$$y_{n+1} = y_n + h(5y'_{n+\frac{1}{2}-\beta} + 8y'_{n+\frac{1}{2}} + 5y_{n+\frac{1}{2}+\beta})/18, \qquad (23)$$

where $\beta = \sqrt{5}/10$. For calculating the values of $y_{n+\frac{1}{2}+\beta}$ we can use formulae (22) for $\alpha = 1/2 \pm \sqrt{15}/10$. However, in this case for determining the values of the y_{n+1} and $y_{n+\frac{3}{2}}$, participating in formula (22) we use the above-reduced scheme.

III. CONCLUSION

As is know, to find solution of the integro-differential equations can be used different ways. It is obvious that each of these methods has its advantages and shortcomings. We are here reviewed and compared some of them. The result obtained, that the method of degenerate kernel provides more accurate results. Therefore, to solve the integro-differential equations, it is desirable to use hybrid methods in conjunction with degenerate kernels. However, the proposed algorithm, we consider the case of degenerate kernels, since in this algorithm, investigation the general case.

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