Presentation of Uncertain Information with Help of Canonically Conjugate Fuzzy Subsets

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Abstract— ***In*** **modern world we mostly deal with two type uncertainty based on vagueness and ambiguity accordingly. Modeling situation with first type uncertainty mostly based on Fuzzy sets and degree of fuzziness. Major inconvenience with fuzzy modeling is based on expert estimations, we'll need additional criteria to choose "right" expert, with "right" estimations. When there is not enough information on property we'd like to model, we are offering to find canonically conjugate one that would be easier to describe and is completing the information about object. We will construct model that would be "mean"-ed: based on expert estimation, canonically conjugate property and optimality condition.**

Keywords— *fuzzy information; uncertainty; optimal model; fuzzy differential equations*

I. INTRODUCTION

In the space of incomplete and uncertain information the ability of correct decision-making is the most incredible feature of human intellect, the modeling of the human mind principle and using it in the new generation computer systems, is the one of the main task of scientists. The significant step ahead in this direction was made 30 years ago by the California University Professor, L. Zadeh. His work "Fuzzy Subsets" [2] was the base of the intellectual system modeling and became the initial point of new mathematical theory.

Zadeh made the generalization of set, as the classical Cantor notion, accepting that the characteristic function (membership function) can have any meaning in the internal [0,1] and named such sets "fuzzy". He also explained the set of operations on fuzzy sets and generalized the famous logical methods such as "modus ponens" and "modus tollens".

Introducing the linguistic variable notion and accepting the existence of fuzzy subsets as its meaning Zadeh created the powerful system of the intellectual processes, fuzziness and uncertainties description. The further works of the Professor Zadeh and his followers [3] have created the important base for the fuzzy control methods in engineering, industrial practice.

Today two sciences systematically study uncertainty:

Fundamental science physics is the leading in study of the material - physical world. There are two types of uncertainties connected with this science: probabilistic (objective) connected with deficiency in empiric information, received by the observation and virtual - intrinsic, quantum, placed in object directly by nature or is the result of the deficiency in means of description language.

Fundamental science of informatics is the leading in study of the non-material (informational) world, one of the demonstrations of which is the uncertainty connected with the ambiguity and fuzziness. Uncertainty is intrinsic to the expert estimation and to natural language [4], as the means of description informational model.

Connection between these sciences seems to be natural as information presentation is done in terms of material world

II. THEORY OF COLOR

Let, is given set Ω (universal set) and defined property $\wp$ in it. Let us note by $\wp_1(\Omega) and \wp_2(\Omega)$ subset of Ω, defined by elements $\omega \in \Omega$ for which the expression $\wp[\omega]$ (ω possesses color $\wp$) is true or false appropriately. Further let $\wp_0(\Omega) \subseteq \wp_{\neq 1}(\Omega)$. We can consider color $\neg \wp$ defined in Ω:

$$\neg \wp[\omega] \Leftrightarrow \omega \in \wp_0(\Omega)$$

If $\overline{\wp}$ defines color complementary to $\wp$ then in Ω following relation has place:

$$\neg \wp[\omega] \Rightarrow \overline{\wp}[\omega]$$

Opposite implication is fair only on set: $A(\Omega) = \wp_1(\Omega) \cup \wp_0(\Omega)$. With the help of $\wp_0$, it's possible to define such various $\neg \wp$ that if $\neg \wp$ is true, then $\wp$ is false. But opposite implication has place only on set $A(\Omega) \subseteq \Omega$.

Let's check how it's possible to construct set $\wp_0(\Omega) \subseteq \wp_{\neq 1}(\Omega)$. With this aim, let's assume that each elements of Ω can possess color $\wp$ in different amount. Further let's consider that we are able to attach its compatibility measure with color $\wp$ to each element $\omega \in \Omega$. Formally there is given such a reflection:

$$\mu_{\wp} : \Omega \to [0,1]$$

that:

$$\wp[\omega] \Leftrightarrow (\mu_{\wp}(\omega) = 1)$$

For each $\omega \in \Omega$, $\mu_{\wp}(\omega)$ is called the value of membership function of ω with $\wp$ or membership measure of ω to $\wp_1(\Omega)$. If $\mu_{\wp}(\omega) = 1$ we will say that ω possesses color $\wp$ .If $\mu_{\wp}(\omega) = 0$ then ω does not possess color $\wp$. Further $\wp_0(\Omega)$ identify with the subset of $\wp_{\neq 1}(\Omega)$ elements, not possessing color $\wp$. Color $\wp$ in Ω satisfying the described above-mentioned conditions we will call "measurable" in Ω. If additionally assume that $\wp_1(\Omega)$ is not empty, $\wp$ we will call "completely measurable".

Let's assume that color $\wp$ is characterized by numerical parameter ξ.

Main assumption: ξ numerical characteristic of color is the random quantity [5]. In the referent system Ω is hidden parameter.

Let's distribution of probabilistic values $\xi_{\wp}(x_{\omega})$ ($\in \Re$) is characterized by density $\rho_{\wp}(x_{\omega})$,

$$\int_{\Re} \rho_{\wp}(x_{\omega}) dx = 1 \tag{1}$$

Quantity

$$x_{\omega}^* = M\xi_{\wp} = \int_{\Re} x\rho_{\wp}(x_{\omega}) dx \tag{2}$$

call calculated value of color $\wp$ in $\omega \in \Omega$.

Note, that formula (5) satisfies relation between set of calculated values X^* and universal set Ω, that is why the following definitions are clear:

$$\begin{aligned}
&(\text{Set } x_{\omega}^*, \text{ where } x_{\omega}^* \in X^* \text{ and } \omega \in \wp(\Omega)) \equiv \wp(\Re),\\
&(\text{Set } x_{\omega}^*, \text{ where } x_{\omega}^* \in X^* \text{ and } \omega \in \wp_1(\Omega)) \equiv \wp_1(\Re),\\
&(\text{Set } x_{\omega}^*, \text{ where } x_{\omega}^* \in X^* \text{ and } \omega \in \wp_0(\Omega)) \equiv \wp_0(\Re).
\end{aligned} \tag{3}$$

We transferred uncertainty structure (system) Ω in $\Re$.

If $\wp_1(\Omega)$ is non-empty set, exist such ω, that $\int_{\wp_1(\Re)} \rho_{\wp}(x_{\omega}) dx = 1$.

Except of $M\xi$, presence of color to ω is characterized by dispersion also:

$$\sigma_{\wp}^2(\omega) \equiv D(\xi_{\wp}) = \int_{\Re} (x_{\omega} - x_{\omega}^*)\rho_{\wp}(x_{\omega}) dx_{\omega} \tag{4}$$

In our model exactly $\sigma_{\wp}^2(\omega)$ is connected with definition of presence $\wp$ color to ω. If $\sigma_{\wp}^2(\omega) \to 0$, we'll say $\wp$ has quite define value ω. The more $\sigma_{\wp}^2(\omega)$ is the uncertain $\wp$ in ω. If $\sigma_{\wp}^2(\omega) \to \infty$ it means ω has no $\wp$ color. Thus, if $\mu_{\wp}(\omega) = 1$, we will say that x_{ω}^* possesses color $\wp$, if $\mu_{\wp}(\omega) = 0$, than x_{ω}^* does not possess color $\wp$.

Definition 1: For $\forall \omega \in \Omega$ let's introduce some interval of $\wp$ values with the help of relation:

$$\mu_{\tilde{\wp}}(\omega) = \int_{II} \rho_{\wp}(\omega) dx_{\omega} = \int_{\Re} II_{\wp}(\omega)\rho_{\wp}(\omega) dx_{\omega} \tag{5}$$

Where $II_{\wp}(\omega)$ is defined by expert in such manner that $\mu_{\tilde{\wp}}(\omega)$ should be membership function of ω with color $\wp$. Let's call interval defined by (5) as the characteristic interval of color $\wp$.

Definition 2: informational function of color $\wp$ let's call the following expression:

$$|x_{\omega}; \wp\rangle = \sqrt{\rho_{\wp}(x_{\omega})} e^{i\varphi_{\wp}} \tag{6}$$

Where $\varphi_{\wp}$ is random phase and it is a real quantity. We took the advantage of Dirac [6] nomenclature. We will use this function for the presentation of the information (uncertainty) contained in color $\wp$. Informational function module square defines membership function (precisely the appropriate density):

$$\rho_{\wp}(x_{\omega}) = \langle x_{\omega}; \wp | x_{\omega} \wp \rangle \tag{7}$$

Let's $|x_{\omega}; \wp\rangle \in L^2(\Re)$ (Hilbert space). Let's consider the Fourier transformation of this ket-vector:

$$\hat{F}|x_{\omega}; \wp\rangle = \frac{1}{\sqrt{2\pi c}} \int_{\Re} |x_{\omega}; \wp\rangle \exp\left(-\frac{i}{c} x_{\omega} x_{\omega c}\right) dx_{\omega} \tag{8}$$

Where C is constant.

Expression (8) is identified with informational function in x_c presence:

$$\hat{F}|x_{\omega}; \wp\rangle = |x_{c\omega}; \wp_c\rangle \tag{9}$$

Where $\wp_c$ is canonically conjugated in relation to $\wp$ color. Fuzzy subset – canonically conjugated in relation to $\tilde{\wp}$ is appropriate to this color $\tilde{\wp}_c$, membership function of which is defined by the following formula:

$$\mu_{\wp_c}(\omega_c) = \int_{II_{\wp_c}(\omega_c)} \langle x_{c\omega};\wp_c | x_{c\omega};\wp_c\rangle dx_{c\omega} = = \int_{\Re} II_{\wp_c(\omega_c)} \langle x_{c\omega};\wp_c | x_{c\omega};\wp_c\rangle dx_{c\omega} \quad (10)$$

(See formula (5).)

III. The Color Operators

Let us consider some attribute $\wp$ (color) characterizing system mode (state). Mathematical apparatus, that mirror the impact of measurements or expert estimation on a system mode (state), is allowing us to calculate the observed (estimated) property $\wp$ of an object in any state $\left|x^*_\omega;\wp\right\rangle$ [7]. In space of information functions operator of color $\hat{\wp}$ is appropriate to color $\wp$. Eigenvectors of this operator are $\left|x_\omega;\wp\right\rangle$ state vectors, where the $\wp$ property is taking the x_{ω_n} defined values. They are the eigenvalues of this operator:

$$\hat{\wp}\left|x_{\omega_n};\wp\right\rangle = x_{\omega_n}\left|x_{\omega_n};\wp\right\rangle \quad (11)$$

The representation of observed $\wp$ property with help of linear operator that satisfies (11), is convenient also because, $\hat{\wp}$ operator is transforming some $\left|x_\omega;\wp\right\rangle$ vector to another $\left|x'_\omega;\wp\right\rangle$ vector:

$$\hat{\wp}\left|x_\omega;\wp\right\rangle = \left|x'_\omega;\wp\right\rangle \quad (12)$$

So the projection of $\left|x'_\omega;\wp\right\rangle$ vector on $\left|x_\omega;\wp\right\rangle$ is the $\xi_\wp$ mathematical expectation in state $\left|x_\omega;\wp\right\rangle$:

$$x^*_\omega \equiv \langle\hat{\wp}\rangle \equiv M\xi_\wp = \langle x_\omega;\wp | x'_\omega;\wp\rangle = \langle x_\omega;\wp|\hat{\wp}|x_\omega;\wp\rangle \quad (13)$$

Analogically

$$\hat{\wp}_c\left|x_{c\omega_n};\wp_c\right\rangle = x_{c\omega_n}\left|x_{c\omega_n};\wp_c\right\rangle \quad (14)$$

$$\hat{\wp}_c\left|x_{c\omega};\wp_c\right\rangle = \left|x'_{c\omega};\wp_c\right\rangle \quad (15)$$

$$x^*_{c\omega} \equiv \langle\hat{\wp}_c\rangle \equiv M\xi_{\wp_c} = \langle x_{c\omega};\wp_c | x'_{c\omega};\wp_c\rangle = = \langle x_{c\omega};\wp_c|\hat{\wp}_c|x_{c\omega};\wp_c\rangle \quad (16)$$

Operators $\hat{\wp}$ and $\hat{\wp}_c$ are connected with the following commutation:

$$\hat{\wp}\,\hat{\wp}_c - \hat{\wp}_c\hat{\wp} = ic\hat{E} \quad (17)$$

Where $\hat{E}$ is operator of identity presence. This relation should define the quantity connection between canonically conjugated colors. Formula (17) bounds the simultaneous calculation of canonically conjugated colors. The uncertainty principle, analogical to the Heisenberg's principle, that allows introduction of definitely optimal fuzzy subset $\tilde{\wp}$, for $\wp$ and. $\wp_c$. The complete system of proper vectors satisfies the completeness condition:

$$\int dx_\omega|\wp\rangle\langle\wp| = \hat{I}\ , \quad (18)$$

$$\int dx_{\omega c}|\wp_c\rangle\langle\wp_c| = \hat{I} \quad (19)$$

Where $\hat{I}$ is unique operator in Hilbert's space.

The following relations occur:

$$\hat{\wp}_c\left|x_\omega;\wp\right\rangle = -ic\frac{d}{dx_\omega}\left|x_\omega;\wp\right\rangle \quad (20)$$

$$\hat{\wp}\left|x_{c\omega};\wp_c\right\rangle = ic\frac{d}{dx_{c\omega}}\left|x_{c\omega};\wp_c\right\rangle \quad (21)$$

Connection between canonically conjugate colors is defined by:

$$\sigma^2_\wp\sigma^2_{\wp_c} \geq \frac{c^2}{4} \quad (22)$$

Where

$$\sigma^2_\wp = \mu_{\tilde{\wp}}\sigma^2_\wp + \mu_{\neg\tilde{\wp}}\sigma^2_{\neg\wp} \quad (23)$$

$$\sigma^2_{\wp_c} = \mu_{\tilde{\wp}_c}\sigma^2_{\wp_c} + \mu_{\neg\tilde{\wp}_c}\sigma^2_{\neg\wp_c} \quad (24)$$

IV. Probabilistic Model of Fuzzy Subset and its Canonically Conjugate One

Main Definition: (2), (3), (5) equations define the following set [8]:

$$\tilde{\Omega} = \{\tilde{\omega} \equiv (\omega;\mu_\wp(\omega)) : \omega \in \Omega\} \quad (25)$$

Call $\tilde{\Omega}$, the probabilistic model of Ω universal set's fuzzy subset. Analogically defined subset, on the basis of assumption 1. and relations defined by formulas: (9),(10),(16),(24) we will call $\tilde{\Omega}_c$ probabilistic model of canonically conjugate one.

Both $\tilde{\Omega}$ and $\tilde{\Omega}_c$ subsets consist information on state of ω, and these information are completing each other.

V. Characteristic Functions of Canonically Conjugate Colors

Let us consider the operators:

$$\hat{M}(\alpha) = \exp(i\alpha\hat{\wp})$$

and

$$\hat{M}(\beta) = \exp(i\beta\hat{\wp}_c) \qquad (26)$$

The scalar products:

$$M(\alpha) = \langle x_\omega;\wp|\hat{M}(\alpha)|x_\omega;\wp\rangle$$

and

$$M_c(\beta) = \langle x_\omega;\wp|\hat{M}(\beta)|x_\omega;\wp\rangle \qquad (27)$$

are characteristic functions of canonically conjugate colors $\wp$ and $\wp_c$ appropriately.

In case of two fuzzy subset Cartesian product's membership function calculation, we'll use the appropriate characteristic functions [9].

It is not hard to calculate $\hat{\wp}$ and $\hat{\wp}_c$ commutative color operator's characteristic functions:

$$\hat{M}(\alpha_1,...,\alpha_n) = \exp\left(i\sum_{k=1}^{n}\alpha_k\hat{\wp}_k\right),$$

$$M(\alpha_1,...,\alpha_n) = \\ = \langle x_{\omega_1},...,x_{\omega_n};\wp_1,...,\wp_n|\hat{M}(\alpha_1,...,\alpha_n)|x_{\omega_1},...,x_{\omega_n};\wp_1,...,\wp_n\rangle, \qquad (28)$$

$$\rho_{\wp_1\times...\times\wp_n}(x_{\omega_1},...,x_{\omega_n}) = \\ = \frac{1}{(2\pi)^n}\int_{\Re^n} M(\alpha_1,...,\alpha_n)\exp\left(-i\sum_{k=1}^{n}\alpha_k\omega_k\right)\prod_{i=1}^{n}d\alpha_i$$

For non-commutative colors $\wp$ and $\wp_c$ there are possible different formulations, in frames of which the information functions in phase space might be bounded with information state of color which is estimated by expert. Detailed formalism is based on Weil transformation and Wigner function.

If we will take characteristic function operator instead of main operator $\hat{A}$, we will receive:

$$W_{\wp\times\wp_c}(x_\omega,x_{\omega_c}) = \\ = \frac{1}{2\pi}\int\langle x_\omega - \pi c v|\exp(-ivx_{c\omega})|x_\omega + \pi c v\rangle dv \qquad (29)$$

$$W_{\wp\times\wp_c}(x_\omega,x_{\omega_c}) = \\ = \frac{1}{\sqrt{2\pi c}}\exp\left(-i\pi c\frac{\partial^2}{\partial x_\omega\partial x_{c\omega}}\right)\langle x_\omega;\wp|x_{c\omega};\wp_c\rangle\times \\ \times\exp\left(\frac{i}{2\pi c}x_\omega x_{c\omega}\right) \qquad (30)$$

VI. THE OPTIMAL FUZZY SUBSETS AND OPTIMAL MEMBERSHIP FUNCTION

Definition: Fuzzy subsets, that corresponding to the minimal value of $\sigma^2_{\wp}\sigma^2_{\wp_c}$ product, call Optimal.

Optimal information functions have the view:

$$|x_\omega;\wp\rangle = \frac{1}{\sqrt[4]{2\pi\sigma^2_\wp}}\exp\left(-\frac{(x_\omega - x^*_\omega)^2}{4\sigma^2_\wp} + \frac{i}{c}x^*_{c\omega}x_\omega\right) \qquad (31)$$

$$|x_{c\omega};\wp\rangle = \frac{1}{\sqrt[4]{2\pi\sigma^2_{\wp_c}}}\exp\left(-\frac{(x_{c\omega} - x^*_{c\omega})^2}{4\sigma^2_{\wp_c}} + \frac{i}{c}x^*_{\omega}x_{c\omega}\right) \qquad (32)$$

In this case Wigner function will have the view:

$$W^{opt}_{\wp\times\wp_c}(x_\omega,x_{c\omega}) = \\ = (2\pi c)^{-1}\exp\left(-\frac{(x_\omega - x^*_\omega)^2}{2\sigma^2_{\wp_c}} - \frac{2(x_{c\omega} - x^*_{c\omega})^2\sigma^2_\wp}{c^2}\right)$$

For considered optimal model, the membership function, appropriate to canonically conjugate calibration [10], is presented by the following formula:

$$\mu_\wp(\omega) = \frac{1}{\sqrt{2\pi\sigma^2_\wp}}\int_{x^*_\omega-\alpha(\omega)\sigma_\wp}^{x^*_\omega+\alpha(\omega)\sigma_\wp}\exp\left(-\frac{(x_\omega - x^*_\omega)^2}{2\sigma^2_\wp}\right)dx_\omega = \\ = \Phi\left(\frac{\alpha(\omega)}{\sqrt{2}}\right) \qquad (33)$$

Analogically we will have:

$$\mu_{\wp_c}(\omega_c) = \Phi\left(\frac{\alpha_c(\omega_c)}{\sqrt{2}}\right) \qquad (34)$$

And the corresponding optimal membership function will be:

$$\mu^{opt}_{\wp\times\wp_c}(\omega,\omega_c) = \iint_{I_\varphi\times I_{\varphi_c}} W^{opt}_{\varphi\times\varphi_c}(x_\omega,x_{c\omega})dx_\omega dx_{c\omega}$$

With characteristic Intervals

$$I_\varphi = [x^*_\omega - \alpha_1(\omega)\sigma_\varphi; x^*_\omega + \alpha_2(\omega)\sigma_\varphi]$$

$$I_{\varphi_c} = \left[x^*_{c\omega} - \alpha_{1c}(\omega)\sigma_{\varphi_c} ; x^*_{c\omega} + \alpha_{2c}(\omega)\sigma_{\varphi_c} \right],$$

where $\alpha_1(\omega)$, $\alpha_2(\omega)$, $\alpha_{1c}(\omega)$ and $\alpha_{2c}(\omega) \in \Re$ and are determined by the structure of supports of canonically conjugate fuzzy subsets.

VII. CONCLUSION

The development of decision-making computer systems, based on fuzzy logics, has reached its peak lately. It was appropriate to work on data with the probabilistic-statistic methods because the classical methods do not give the convincing result in this case. The compilation of fuzzy information as well as the effective and fast realizable algorithm development - is very actual task in modern world. The works of the famous scientist - L. Zadeh, D. Dubois, H. Prade, A. Kandel, A. Kauffman and others are dedicated to this issue.

The work is the new attempt of fast realizable algorithm development for optimal compilation of fuzzy information.

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