SOME SPECTRAL PROPERTIES OF A NEW GENERALIZED DIFFERENCE OPERATOR

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The main purpose of this paper is to determine the spectrum of a new generalized difference operator, denoted by $\Delta_{a,b}$, over the sequence space c_0. The norm of the operator $\Delta_{a,b}$ on the sequence space c_0 has been found. The results of this paper generalize the corresponding results of [3], [4] and [7].

1. Introduction

By $B(X)$, we denote the set of all bounded linear operators on the Banach space X *into* itself. Let $X \neq \varnothing$ be a complex normed space and consider a linear operator $T : D(T) \to X$, with domain $D(T) \subseteq X$. With T we associate the operator

$$T_\lambda = T - \lambda I,$$

where λ is a complex number and I is the identity operator on $D(T)$. By a regular value λ of T we mean a complex number such that

(**R1**) T_λ^{-1} exists,

(**R2**) T_λ^{-1} is bounded,

(**R3**) T_λ^{-1} is defined on a set which is dense in X.

The *resolvent set* of T, denoted by $\rho(T, X)$, is the set of all regular values λ of T. Its compliment $\sigma(T, X) = \mathbb{C} \setminus \rho(T, X)$ in the complex plane $\mathbb{C}$ is called the spectrum of T. Furthermore, the spectrum $\sigma(T, X)$ is partitioned into three disjoint sets as follows:

The *point* (*discrete*) *spectrum* $\sigma_p(T, X)$ is the set such that T_λ^{-1} does not exist. Any such $\lambda \in \sigma_p(T, X)$ is called an *eigenvalue* of T.

The *continuous spectrum* $\sigma_c(T, X)$ is the set such that T_λ^{-1} exists and satisfies (**R3**) but not (**R2**), that is T_λ^{-1} is unbounded.

The *residual spectrum* $\sigma_r(T, X)$ is the set such that T_λ^{-1} exists (and may be bounded or not) but not satisfy (**R3**), that is the domain of T_λ^{-1} is not dense in X.

We summarize the knowledge in the existing literature concerning with the spectrum of the linear operator defined by some particular limitation matrices over some sequence spaces. The fine spectrum of the difference operator Δ over the sequence space l_p, $(p \geq 1)$ is determined by A. Akhmedov and F. Başar [1] and over the sequence space c_0 and c by B. Altay and F. Başar [3]. B. De Malafosse [6] computed the spectrum of the difference operator on the space s_r. A. Akhmedov and F. Başar [2] determined the fine spectrum of the difference operator on the space bv_p, $(1 \leq p < \infty)$. Note that the sequence space bv_p was introduced and studied by B. Altay and F. Başar [5]. The continuous dual of bv_p determined by A. Akhmedov in [2].

We introduce the generalized difference operator $\Delta_{a,b}$ on the sequence space c_0 as follows:

$\Delta_{a,b} : c_o \to c_o$ is defined by,

$$\Delta_{a,b}x = \Delta_{a,b}(x_n) = (b_{n-1}x_{n-1} + a_n x_n)_{n=0}^{\infty} \text{ with } x_{-1} = 0, b_{-1} = 0$$

where (a_n) and (b_n) are two sequences of nonzero real numbers such that:

$$\lim_{n\to\infty} a_n = a \ , \ \sup_n |a_n| = A, \ \lim_{n\to\infty} b_n = b \neq 0, \ \sup_n |b_n| = B \ and \ a_n \neq a+b, \ \ a_n \neq a-b \ \text{for all } n \in \mathbb{N} \ .$$

Lemma 1. ([8, pp. 129]). *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(c_0)$ from c_0 to itself if and only if*
(1) *the rows of A in l_1 and their l_1 norms are bounded,*
(2) *the columns of A are in c_0.*
The operator norm of T is the supremum of the l_1 norms of the rows.

In this paper we determine the spectrum of the generalized difference operator $\Delta_{a,b}$ on the sequence space c_0. The results of our paper not only generalize the corresponding results of [3], [4] and [7] but also give results for some more operators.

2. The spectrum of the operator $\Delta_{a,b}$ on the sequence space c_0

In this section, we establish the boundedness of the operator $\Delta_{a,b}$ on c_0. Also, we compute the spectrum and the point spectrum of the operator $\Delta_{a,b}$ on the sequence space c_0.

Theorem 1. $\Delta_{a,b} \in B(c_0)$ with a norm $\|\Delta_{a,b}\|_{c_0} = \sup_k (|a_k| + |b_{k-1}|)$.

Proof. Proof is simple. So we omit it. □

By $\sigma(\Delta_{a,b}, c_0)$ we denote the spectrum of $\Delta_{a,b}$. The main result of this paper is

Theorem 2. Denote the set $\{\lambda \in \mathbb{C} : |\lambda - a| \leq |b|\}$ by D and the set $\{a_k : a_k \notin D\}$ by E. Then $\sigma(\Delta_{a,b}, c_0) = D \cup E$.

Proof. Let $\lambda \notin D \cup E$ and let $y = (y_k) \in c_0$. Then $|\lambda - a| > |b|$ and $\lambda \neq a_k$, for all $k \in \mathbb{N}$. By solving the equation $(\Delta_{a,b} - \lambda I)x = y$, for $x = (x_k)$ in terms of y, we get

$$x_k = \frac{(-1)^k b_0 b_1 \ldots b_{k-1}}{(a_0-\lambda)(a_1-\lambda)\ldots(a_k-\lambda)} y_0 + \ldots - \frac{b_{k-1}}{(a_{k-1}-\lambda)(a_k-\lambda)} y_{k-1} + \frac{1}{(a_k-\lambda)} y_k \qquad , k \in \mathbb{N}$$

Then,

$$\left(\Delta_{a,b} - \lambda I\right)^{-1} = (s_{nk}) = \begin{pmatrix} \frac{1}{(a_0-\lambda)} & 0 & 0 & \cdots \\ \frac{-b_0}{(a_0-\lambda)(a_1-\lambda)} & \frac{1}{(a_1-\lambda)} & 0 & \cdots \\ \frac{b_0 b_1}{(a_0-\lambda)(a_1-\lambda)(a_2-\lambda)} & \frac{-b_1}{(a_1-\lambda)(a_2-\lambda)} & \frac{1}{(a_2-\lambda)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Let $S_n = \sum_{k=0}^{\infty} |s_{nk}|$. Clearly, for each $n \in \mathbb{N}$, the series S_n is convergent since it is finite. Next, we prove that $sup\, S_n$ is finite.

Since $\lim_{k\to\infty}\frac{|b_k|}{|a_k-\lambda|}=\frac{|b|}{|a-\lambda|}=q<1$. Then there exists $k_0\in\mathbb{N}$ and $q_0 < 1$ such that $\frac{|b_k|}{|a_k-\lambda|}<q_0<1$ for all $k\ge k_0+1$. Then, for each $n\in\mathbb{N}$, we can prove that $S_n\le A\left(1+q_0+q_0^2+...+q_0^{n-k_0-1}\right)\frac{1}{|a_n-\lambda|}\le A\left(1+q_0+q_0^2+...\right)\frac{1}{|a_n-\lambda|}=\frac{A}{1-q_0}.\frac{1}{|a_n-\lambda|}$, where $A=\max_{0\le i\le k_0}\{m_i\}$, $m_i=\frac{|b_i||b_{i+1}|...|b_{k_0}|}{|a_i-\lambda||a_{i+1}-\lambda|...|a_{k_o}-\lambda|}$, $i=0,1,2,...,k_0$. But, for large *n*, we have $\frac{1}{|a_n-\lambda|}<q_1<\frac{1}{|b|}$, and so $S_n\le\frac{Aq_1}{1-q_0}$. Thus $sup\,S_n<\infty$.

Now it is easy to prove that $\lim_{n\to\infty}|s_{nk}|=0$, for all $k\in\mathbb{N}$ and so the columns of $\left(\Delta_{a,b}-\lambda I\right)^{-1}$ are in c_0. Then, from Lemma 1, we have $(\Delta_{a,b}-\lambda I)^{-1}\in B(c_0)$ and so $\lambda\notin\sigma(\Delta_{a,b},c_0)$. Thus $\sigma(\Delta_{a,b},c_0)\subseteq D\cup E$

Conversely, let $\lambda\notin\sigma(\Delta_{a,b},c_0)$. Then $(\Delta_{a,b}-\lambda I)^{-1}\in B(c_0)$ and hence $(\Delta_{a,b}-\lambda I)^{-1}e_1$ existed in c_0, where e_1 is the unit sequence (1, 0, 0, 0, …). Then we can easily see that $\lim_{k\to\infty}\left|\frac{b_k}{a_{k+1}-\lambda}\right|=\left|\frac{b}{a-\lambda}\right|\le 1$ and $\lambda\ne a_k$, for all $k\in\mathbb{N}$. Then $\{\lambda\in\mathbb{C}:|\lambda-a|<|b|\}\subseteq\sigma(\Delta_{a,b},c_0)$ and $\{a_k:k\in\mathbb{N}\}\subseteq\sigma(\Delta_{a,b},c_0)$. But, $\sigma(\Delta_{a,b},c_0)$ is a compact set, and so it is closed. Then $D=\{\lambda\in\mathbb{C}:|\lambda-a|\le|b|\}\subseteq\sigma(\Delta_v,c)$ and $E=\{a_k:a_k\notin D\}\subseteq\sigma(\Delta_{a,b},c_0)$. This completes the proof. □

The point spectrum of the operator $\Delta_{a,b}$ is given by the following theorem

Theorem 3. $\sigma_p(\Delta_{a,b},c_0)=\begin{cases}E, & \textit{if there exists } m\in\mathbb{N}: a_i\ne a_j\ \forall i,j\ge m\\ \varnothing, & \textit{otherwise}\end{cases}$

Proof. Consider the equation $\Delta_{a,b}x=\lambda x$ for $x\ne\theta=(0,0,0,...)$ in c_0. Then

$$(a_0-\lambda)x_0=0 \text{ and } (a_k-\lambda)x_k+b_{k-1}x_{k-1}=0, \text{ for all } k=1,2,3,....$$

Hence, for all $\lambda\notin\{a_k:k\in\mathbb{N}\}$, we have $x_k=0$, for all $k\in\mathbb{N}$, which is a contradiction. So, $\lambda\notin\sigma_p(\Delta_{a,b},c_0)$. This shows that $\sigma_p(\Delta_{a,b},c_0)\subseteq\{a_k:k\in\mathbb{N}\}$.

Now, if $\lambda=a_i$ and there exists $j\in\mathbb{N}$, $j>i$ such that $a_i=a_j$, then we can easily see that $x_k=0$ for all $k<j$. Then we have the following cases:

Case (i): Let (a_k) is such that $a_i\ne a_j$ for all $i,j\in\mathbb{N}$ and let, $\lambda=a_0$. If $x_0=0$, then $x_k=0$ for all $k\in\mathbb{N}$ and so we have a contradiction as $x\ne\theta$. Also, if $x_0\ne 0$ then $x_{k+1}=\frac{-b_k}{a_{k+1}-a_0}x_k$ for all $k=0,1,2,...$, and hence

$$\lim_{k\to\infty}\left|\frac{x_{k+1}}{x_k}\right| = \left|\frac{b}{a-a_o}\right|.$$

But $\left|\frac{b}{a-a_o}\right| \neq 1$, since $a_0 \neq a+b,\ a_0 \neq a-b$. Then, $x \in c_0$ if and only if $|a-a_0| > |b|$. Then $a_0 \in \sigma_p(\Delta_{a,b}, c_0)$ if and only if $|a-a_0| > |b|$.

Similarly, we can prove that $a_k \in \sigma_p(\Delta_{a,b}, c_0)$ if and only if $|a-a_k| > |b|$. Thus $\sigma_p(\Delta_{a,b}, c_0) = E$ in this case.

Case (ii): If (a_k) is such that there exists $m \in \mathbb{N}$ with $a_i \neq a_j$ for all $i, j \geq m$, then we can prove, as in Case (i), that $a_k \in \sigma_p(\Delta_{a,b}, c_0)$ if and only if $|a-a_k| > |b|$. Thus $\sigma_p(\Delta_{a,b}, c_0) = E$.

Case (iii): If (a_k) is not as in Case (i) or Case (ii), that is for all $m \in \mathbb{N}$ there exists $i < m$ and $j \geq m$ such that $a_i = a_j$, then we must have $x = \theta$ which is a contradiction. Thus $\sigma_p(\Delta_{a,b}, c_0) = \emptyset$ in this case.

This completes the proof. $\square$

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