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DIFFERENCE SCHEME OF HIGHER ACCURACY ORDER FOR SOLUTION A NONLOCAL PROBLEM

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Let us denote through Π a rectangle with vertices (0,0), (1,0), (1,b), (0,b), where b-rational number. Let Γ-boundary of this rectangle.

We introduce net square by lines $x = x_i = ih, \quad y = y_j = jh \ (i = 0,1,...,1/h, \ j = 0,1,...,b/h)$, where $1/h$ and b/h-integer numbers. Let

$$\Pi_h = \{(x, y) : x = x_i = ih, i = 0,1,...,1/h, y = y_j = jh, j = 0,1,...,b/h\},$$

and Γ_h - set of net knots, lying on Γ.

Consider the following nonlocal problem

$$\left.\begin{array}{l} \Delta u = 0 \ in \ \Pi, \\ u(x,0) = u(x,b) = 0 \ (0 < x < 1), \\ u(1, y) = \varphi(y) \quad (0 < y < b), \\ u(0, y) = \alpha u(c, y) + f(y) \ \ (0 < y < b, 0 < c < 1, \alpha \geq 0) \end{array}\right\} \tag{1}$$

where $\varphi(y), f(y)$ are five times continuously differentiable functions and

$$\varphi(0) = \varphi(b) = 0,$$
$$f(0) = f(b) = 0.$$

Evidently, truncation error may be represented as the sum of truncation errors of following problems:

$$\begin{cases} \Delta u = 0, \\ u(x,0) = 0, \\ u(x,b) = 0, \\ u(1, y) = 0, \\ u(0, y) = \alpha u(c, y) + f(y), \end{cases} \qquad \begin{cases} \Delta u = 0, \\ u(1, y) = f(y), \\ u(x,0) = 0, \\ u(x,b) = 0, \\ u(0, y) = \alpha u(c, y) \end{cases}$$

At first we investigate problem

$$\left.\begin{array}{ll} \Delta u = 0 & in \ \ \Pi, \\ u(x,0) = u(x,b) = 0 & (0 < x < 1), \\ u(1, y) = 0, & \\ u(0, y) = \alpha u(c, y) + f(y) \ \ (0 < y < b). & \end{array}\right\} \tag{2}$$

We build corresponding difference scheme in following way:

$$\left.\begin{array}{ll} \Delta_h u_h = 0 & in \ \ \Pi_h, \\ u_h(x,0) = u_h(x,b) = 0 & (0 < x < 1), \\ u_h(1, y) = 0, & \\ u_h(0, y) = \alpha u_h(c, y) + f(y) \ \ (0 < y < b). & \end{array}\right\} \tag{3}$$

Suppose that $x = c$ coincides with on x_i points.

It can be easily verified that the solutions of problems (2) and (3) are defined accordingly by formulas

$$u(x, y) = \sum_{n=1}^{\infty} c_n g(x, n\pi) \sin\frac{n\pi y}{b},$$

$$u_h(x,y)=\sum_{n=1}^{1/h}\gamma_n g(x,\beta_n/h)\sin\frac{n\pi y}{b},$$

where

$$c_n=\frac{2}{b}\int_0^b f(t)\sin\frac{n\pi t}{b}dt,$$

$$\gamma_n=\frac{2h}{b}\sum_{r=1}^{1/h} f(rh)\sin\frac{n\pi rh}{b},$$

$$g(x,z)=\frac{sh(1-x)\frac{z}{b}}{sh\frac{z}{b}-\alpha\, sh(1-c)\frac{z}{b}}$$

and β_n is defined from

$$sh\frac{\beta_n}{2b}=\frac{\sin nh\pi/2b}{\sqrt{1-\frac{2}{3}\sin^2 nh\pi/2b}}. \quad (4)$$

Hence

$$|c_n|\le kn^{-5}, \quad (5)$$

where

$$k=\frac{2b^4}{\pi^5}[|f^{(IV)}(b)|+|f^{(IV)}(0)|]+\frac{4b^5}{\pi^6}\max|f^{(V)}(t)|.$$

It can be easily proved that

$$|\beta_n-nh\pi|\le\frac{(nh\pi)^5}{480b^4}. \quad (6)$$

We have

$$\left|\frac{\partial g(x,z)}{\partial z}\right|\le\frac{1}{16b}(1-\exp(-\frac{8}{3b})-\alpha\exp(-\frac{4}{3b}))^{-2}[x\exp(-\frac{x}{b}z)+\alpha(x+c)\exp(-\frac{x+c}{b}z)],$$

in $1\le n\le 1/h, 0\le y\le b, \sqrt{3}n\pi\ge z\ge\frac{\beta_n}{h}$.

Then, using (6), we have

$$|g(x,\beta_n/h)-g(x,n\pi)|\le\frac{1}{16b}(1-\exp(-\frac{8}{3b})-\alpha\exp(-\frac{4}{3b}))^{-2}\times$$
$$\times[x\exp(-\frac{x}{b}z)+\alpha(x+c)\exp(-\frac{(x+c)}{b}z)]\frac{(n\pi)^5}{480b^4}h^4\le\frac{\pi^5}{7680b^5}(1-\exp(-\frac{8}{3b})-\alpha\exp(-\frac{4}{3b}))^{-2}\times \quad (7)$$
$$\times[x\exp(-\frac{4x}{3b}n)+\alpha(x+c)\exp(-\frac{4(x+c)}{3b}n)]n^5h^4$$

At last not that

$$0\le g(x,z)\le\frac{1}{1-\alpha}\quad(0\le x\le 1,\ z\ge\frac{4b}{3}). \quad (8)$$

Now estimate $|u-u_h|$. We have

$$|u-u_h|\le R_1+R_2,$$

where

$$R_2=\sum_{n=1+1/h}^{\infty}|c_n|g(x,n\pi).$$

It follows from (5) and (8) that

$$R_1 = \sum_{n=1}^{1/h} |c_n| |g(x, \beta_n / h) - g(x, n\pi)|, \ \ R_2 \le K \sum_{n=1+1/h}^{\infty} n^{-5} \le K \frac{h^4}{2}.$$

Using (5) and (7), we received

$$|R_1| \le K \sum_{n=1}^{1/h} n^{-5} [x \exp(-\frac{4nx}{3b}) + \alpha (x+c) \exp(-\frac{4n(x+c)}{3b})](1 - \exp(-\frac{8}{3b}) -$$

$$-\alpha \exp(-\frac{4c}{3b}))^{-2} \frac{\pi^5}{7680 b^5} n^5 h^4 = K \frac{\pi^5}{7680 b^5} h^4 (1 - \exp(-\frac{8}{3b}) - \alpha \exp(-\frac{4c}{3b}))^{-2} \times$$

$$\times \sum_{n=1}^{1/h} [x \exp(-\frac{4nx}{3b}) + \alpha (x+c) \exp(-\frac{4n(x+c)}{3b})] \le$$

$$\le \frac{K \pi^5 h^4}{10240 b^4} (1 - \exp(-\frac{8}{3b}) - \alpha \exp(-\frac{4c}{3b}))^{-2} (1+\alpha),$$

$$|u - u_h| \le K \frac{h^4}{2} + \frac{K\pi^5}{10240 b^4} (1+\alpha)(1 - \exp(-\frac{8}{3b}) - \alpha \exp(-\frac{4c}{3b}))^{-2} h^4,$$

$$|u - u_h| \le K[0{,}5 + \frac{\pi^5}{10240 b^4} (1+\alpha)(1 - \exp(-\frac{8}{3b}) - \alpha \exp(-\frac{4c}{3b}))^{-2}] h^4.$$

Consider the problem (9)

$$\left.\begin{array}{ll} \Delta u = 0 & in \quad \Pi, \\ u(x,0) = u(x,b) = 0 & (0 < x < 1), \\ u(1,y) = \varphi(y), & (0 < y < b), \\ u(0,y) = \alpha u(c,y) & (0 < y < b). \end{array}\right\} \qquad (9)$$

The corresponding difference scheme for this problem will be following

$$\left.\begin{array}{ll} \Delta_h u_h = 0 & in \quad \Pi_h, \\ u_h(x,0) = u_h(x,b) = 0 & , \\ u_h(1,y) = \varphi_h(y), & \\ u_h(0,y) = \alpha u_h(c,y). & \end{array}\right\} \qquad (10)$$

It is easy to prove that the solution of the problem is determined by formulas

$$u(x,y) = \sum_{n=1}^{\infty} c_n g(x, n\pi) \sin \frac{n\pi y}{b},$$

$$u_h(x,y) = \sum_{n=1}^{1/h} \gamma_n g(x, \beta_n / h) \sin \frac{n\pi y}{b},$$

where

$$c_n = \frac{2}{b} \int_0^b \varphi(t) \sin \frac{n\pi t}{b} dt, \quad \gamma_n = \frac{2h}{b} \sum_{r=1}^{1/h} \varphi_h(rh) \sin \frac{n\pi rh}{b},$$

$$g(x,z) = \frac{sh \frac{x}{b} z - \alpha \, sh \frac{(x-c)}{b} z}{sh \frac{z}{b} - \alpha \, sh \frac{(1-c)}{b} z}$$

and β_n is determined from

$$sh \frac{\beta_n}{2b} = \frac{\sin \frac{nh\pi}{2b}}{\sqrt{1 - \frac{2}{3} \sin^2 \frac{nh\pi}{2b}}}. \qquad (11)$$

The solution of difference scheme (10) $u_h(x,y)$ will be taken as approximate solution of problem (9).

Estimate the truncation error of the method. We have

$$\left|u-u_h\right| \le R_1 + R_2 ,$$

where

$$R_1 = \sum_{n=1}^{1/h} \left|c_n\right| \left|g(x,\beta_n/h) - g(x,n\pi)\right|,$$

$$R_2 = \sum_{n=1+1/h}^{\infty} \left|c_n\right| g(x,n\pi).$$

For estimation $\left|u-u_h\right|$:

$$R_1 \le K\frac{1}{b}\left\{\sum_{n=1}^{1/h} n^{-5} n^5 ((a-x)\exp(-\frac{a-x}{b}\frac{4n}{3}) + \alpha(a+c-x)\exp(-\frac{a+c-x}{b}\frac{4n}{3}) + (a-\left|x-c\right|)\times\right.$$
$$\left.\times\exp(-\frac{a-\left|x-c\right|}{b}\frac{4n}{3}) + \alpha(a-\left|x-c\right|)\exp(-\frac{a-\left|x-c\right|}{b}\frac{4n}{3})\right\}\times$$
$$\times(1-\exp(-\frac{8a}{3b}) - \alpha\exp(-\frac{4c}{3b}))^{-2}\frac{\pi^5}{480b^4}h^4 \le\le K\frac{\pi^5}{480b^5}h^4(1-\exp(-\frac{8a}{3b}) -$$
$$-\alpha\exp(-\frac{4c}{3b}))^{-2}\left\{(a-x)\sum_{n=1}^{1/h}(\exp(-\frac{a-x}{b}\frac{4}{3}))^n + \alpha(a+c-x)\sum_{n=1}^{1/h}(\exp(-\frac{a+c-x}{b}\frac{4}{3}))^n +\right.$$
$$\left.+(a-\left|x-c\right|)\sum_{n=1}^{1/h}(\exp(-\frac{a-\left|x-c\right|}{b}\frac{4}{3}))^n + \alpha(a+\left|x-c\right|)\sum_{n=1}^{1/h}(\exp(-\frac{a-\left|x-c\right|}{b}\frac{4}{3}))^n\right\} =$$
$$= K\frac{1}{320b^4}\pi^5(1+\alpha)h^4(1-\exp(-\frac{8a}{3b}) - \alpha\exp(-\frac{4c}{3b}))^{-2}.$$

So

$$\left|u-u_h\right| \le K\frac{h^4}{2} + \frac{1}{320b^4}(1+\alpha)Kh^4\pi^5(1-\exp(-\frac{8a}{3b}) - \alpha\exp(-\frac{4c}{3b}))^{-2},$$

$$\left|u-u_h\right| \le K\left\{0{,}5 + \frac{1}{320b^4}(1+\alpha)\pi^5(1-\exp(-\frac{8a}{3b}) - \alpha\exp(-\frac{4c}{3b}))^{-2}\right\}h^4 .$$