IDENTIFICATION OF TWO UNKNOWN COEFFICIENTS IN PARABOLIC EQUATION

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In work it is considered the two-dimensional inverse problem for the heat equation. It is found the solution of this problem. Also there are defined the unknown coefficients for $U_y(t,x,y)$ and $U(t,x,y)$, depending from t and x. For this aim we are used non-uniform conditions of the redefinition. It is suggested the method of solution the inverse problem based on the theory of difference schemes. The offered method is realized numerically. Also we carried out the experiments on different test functions and different steps of grid.

The equations of parabolic type meet in many sections of mathematics and the mathematical physics. Search of decisions of the differential equations with private derivatives of the second and higher usages always was in sphere of heightened interests of many outstanding mathematicians throughout not so one century.

Many cases when requirements of practice lead to problems of definition of factors of the differential equation (ordinary or in private derivatives) on some known functions from its decision are known. Such problems have received the name of inverse problems of mathematical physics.

Applied importance of inverse problems is so great (they arise in the most various areas of human activity, such as seismology, investigation of minerals, biology, medicine, quality assurance of industrial products etc.) that puts them abreast the most urgent problems of modern mathematics.

As inverse problems for the differential equations are called problems of definition entrance given - factors, the right parts of the differential equations, borders of area, boundary or entry conditions under the additional information on decisions of the equations

The theory and methods of the decision of inverse problems make the important direction of scientific researches in the field of the differential equations in private derivatives. The inverse problem is called as one-dimensional if equation factors depend only on one spatial variable. In a case when the equation depends on several spatial variables, the inverse problem is called as multidimensional.

Problem statement

Let's consider the following problem:

$$U_t(t,x,y) = L(U(t,x,y)) + \mu(t,x)U_y(t,x,y) + \lambda(t,x)U(t,x,y) + F(t,x,y), \quad (1)$$

$$U(0,x,y) = U_0(t,x,y),\ x \in [x_l, x_r],\ y \in [y_l, y_r], \quad (2)$$

$$U(t,x_l,y) = U_1(t,y),\ U(t,x_r,y) = U_2(t,y),\ t \in [0,T],\ y \in [y_l, y_r],$$
$$U(t,x,y_l) = U_3(t,x),\ U(t,x,y_r) = U_4(t,x),\ t \in [0,T],\ x \in [x_l, x_r]. \quad (3)$$

Here $f(t,x,y)$, $U_0(x,y)$, $U_1(x,y)$, $U_2(x,y)$, $U_3(x,y)$, and $U_4(x,y)$ are known functions. The equation (1) – the basic equation, (2) – the entry condition, (3) – boundary conditions, $L(U(t,x,y)) = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}$.

Let's assume also performance of conditions of redefinition::

$$U(t,x,y_1) = U_{y1}(t,x),\ U(t,x,y_2) = U_{y2}(t,x),\ t \in [0,T],\ x \in [x_l, x_r]. \quad (4)$$

Functions $U_{y1}(t,x)$ and $U_{y2}(t,x)$ are known functions.

Problem: to find functions $U(t,x,y)$, $\mu(t,x)$ and $\lambda(t,x)$, satisfying to the equation (1) at $t \in [0,T]$, $x \in [x_l, x_r]$, $y \in [y_l, y_r]$. Function $U(t,x,y)$ also should satisfy to entry conditions (2), boundary conditions (3) and to redefinition conditions (4).

Reduction to a direct problem

Let's put in the equation (1) $y = y_1$. Considering (4), we will receive:

$$(U_{y1})_t(t,x) = L(U(t,x,y_1)) + \mu(t,x)U_y(t,x,y_1) + \lambda(t,x)U(t,x,y_1) + F(t,x,y_1). \qquad (5)$$

After that put in the equation (1) $y = y_2$. Considering (4), we will receive:

$$(U_{y2})_t(t,x) = L(U(t,x,y_2)) + \mu(t,x)U_y(t,x,y_2) + \lambda(t,x)U(t,x,y_2) + F(t,x,y_2) \qquad (6)$$

From (5) and (6) we receive:

$$\mu(t,x) = \frac{VU_{y1} - WU_{y2}}{\Delta}, \ \lambda(t,x) = \frac{WU_y|_{y=y1} - VU_y|_{y=y2}}{\Delta}, \qquad (7)$$

$$\Delta = U_{y1}U_y|_{y=y1} - U_{y2}U_y|_{y=y2},$$
$$V = (U_{y1})_t - L(U)|_{y=y1} - F|_{y=y1}, \quad W = (U_{y2})_t - L(U)|_{y=y2} - F|_{y=y2}. \qquad (8)$$

We substitute the received expressions (7) in the equation (1) and we come to a direct problem:

$$U_t = L(U) + \frac{VU_{y1} - WU_{y2}}{\Delta}U_y + \frac{WU_y|_{y=y1} - VU_y|_{y=y2}}{\Delta}U + F. \qquad (9)$$

Direct problem: to find the function $U(t,x,y)$, satisfying to the equation (9), at $t \in [0,T]$, $x \in [x_l, x_r]$, $y \in [y_l, y_r]$, entry conditions (2), boundary conditions (3) and to conditions of redefinition (4).

Construction of the decision of a direct problem

Let's define a rectangular grid: $\omega_{xy} = ((x_i, y_j) : x_i = ih_x, y_j = jh_y, i = \overline{0, N_x}$, $j = \overline{0, N_y}$, $h_x = (x_r - x_l)/N_x, h_y = (y_r - y_l)/N_y)$. On a piece $t \in [0,T]$ define a grid: $\omega_t = ((t^n) : t^n = n\tau, n = \overline{0, N_t}, \tau = T/N_t)$.

Values of function $U(t,x,y)$ in grid knots we will designate $U_{ij}^n = U(t,x,y)$. Let's construct the differencing scheme for the equation (9), replacing derivatives corresponding differencing schemes:

$$\frac{\partial^2 U}{\partial x^2} \approx \Lambda_x(U_{ij}^n) = \frac{U_{i+1j}^n - 2U_{ij}^n + U_{i-1j}^n}{h_x^2}, \ \frac{\partial^2 U}{\partial y^2} \approx \Lambda_y(U_{ij}^n) = \frac{U_{ij+1}^n - 2U_{ij}^n + U_{ij-1}^n}{h_y^2},$$

$$\frac{\partial U}{\partial y} \approx \frac{U_{ij+1}^n - U_{ij-1}^n}{2h_y}, \ \frac{\partial U}{\partial t} \approx \frac{U_{ij}^{n+1} - U_{ij}^n}{\tau}. \qquad (10)$$

We receive the differencing scheme:

$$\frac{U_{ij}^{n+1} - U_{ij}^n}{\tau} = \sigma(\Lambda_x(U_{ij}^n) + \Lambda_y(U_{ij}^n)) + (1-\sigma)(\Lambda_x(U_{ij}^{n+1}) + \Lambda_y(U_{ij}^{n+1})) +$$

$$+\mu_i^n \frac{U_{ij+1}^n - U_{ij-1}^n}{2h_y} + \lambda_i^n U_{ij}^n + F_{ij}^n, \qquad i=\overline{0,N_x},\ j=\overline{0,N_y},\ n=\overline{0,N_t}, \tag{11}$$

where

$$\mu_i^n = \frac{V_i^n U_{ij_{y1}}^n - W_i^n U_{ij_{y2}}^n}{\Delta_i^n},\ \lambda_i^n = \frac{W_i^n \dfrac{U_{ij_{y1}+1}^n - U_{ij_{y1}-1}^n}{2h_y} - V_i^n \dfrac{U_{ij_{y2}+1}^n - U_{ij_{y2}-1}^n}{2h_y}}{\Delta_i^n}, \tag{12}$$

$i=\overline{0,N_x}$, $j=\overline{0,N_y}$, $n=\overline{0,N_t}$.

Functions Δ_i^n, V_i^n and W_i^n are expressed by parities:

$$\Delta_i^n = U_{ij_{y1}}^n \frac{U_{ij_{y1}+1}^n - U_{ij_{y1}-1}^n}{2h_y} - U_{ij_{y2}}^n \frac{U_{ij_{y2}+1}^n - U_{ij_{y2}-1}^n}{2h_y},\ V_i^n = \frac{U_{ij_{y1}}^{n+1} - U_{ij_{y1}}^n}{\tau} - L(U_{ij_{y1}}^n) - F_{ij_{y1}}^n,$$

$$W_i^n = \frac{U_{ij_{y2}}^{n+1} - U_{ij_{y2}}^n}{\tau} - L(U_{ij_{y2}}^n) - F_{ij_{y2}}^n,\ i=\overline{0,N_x},\ j=\overline{0,N_y},\ n=\overline{0,N_t}. \tag{13}$$

Let's add initial and boundary conditions:

$$U_{ij}^0 = U_0(x_i, y_j),\ i=\overline{0,N_x},\ j=\overline{0,N_y},$$

$$U_{0j}^n = U_1(t^n, y_j),\ n=\overline{0,N_t},\ j=\overline{0,N_y},\ U_{N_xj}^n = U_2(t^n, y_j),\ n=\overline{0,N_t},\ j=\overline{0,N_y},$$

$$U_{i0}^n = U_3(t^n, x_i),\ n=\overline{0,N_t},\ i=\overline{0,N_x},\ U_{iN_y}^n = U_4(t^n, x_i),\ n=\overline{0,N_t},\ i=\overline{0,N_x}, \tag{14}$$

We receive a differencing problem (11) – (14).

The numerical decision

Decision differencing problems (11)-(14) we will find numerically, it is consecutive on layers on time. On each layer on time we receive system of the equations (11).

At $\sigma = 1$ it is received purely obvious scheme, and, expressing U_{ij}^{n+1}, we receive the decision on a layer on time $n+1$ through the decision on a layer n:

$$U_{ij}^{n+1} = \Phi(U_{ij}^n) \tag{15}$$

At $\sigma \neq 1$ it is received the implicit scheme and for a decision finding we use the longitudinal-cross-section scheme constructed under the scheme (11). Main principle of reception longitudinal-cross-section the scheme consists in that the step on time τ breaks on two half step. The first half step is spent from a layer n to the intermediate layer $t = t^n + \tau/2$ which is designated by the semiwhole index $n+1/2$. The second half step is spent from a layer $n+1/2$ to a layer $n+1$.

On the first half step the second derivative on coordinate x was approximated on a layer $n+1/2$, the second derivative on coordinate y was approximated on a layer n, the second derivative on coordinate x was approximated on a layer $n+1/2$, the second derivative on coordinate y was approximated on a layer $n+1$. On half steps it is received systems of equation:

$$\frac{U_{ij}^{n+1/2} - U_{ij}^n}{\tau} = \sigma(\Lambda_x(U_{ij}^n) + \Lambda_y(U_{ij}^n)) + (1-\sigma)(\Lambda_x(U_{ij}^{n+1/2}) + \Lambda_y(U_{ij}^{n+1/2})) +$$

$$+\mu_i^n \frac{U_{ij+1}^n - U_{ij-1}^n}{2h_y} + \lambda_i^n U_{ij}^n + F_{ij}^n, \tag{16}$$

$$\frac{U_{ij}^{n+1}-U_{ij}^{n+1/2}}{\tau}=\sigma(\Lambda_x(U_{ij}^{n+1/2})+\Lambda_y(U_{ij}^{n+1/2}))+(1-\sigma)(\Lambda_x(U_{ij}^{n+1})+\Lambda_y(U_{ij}^{n+1}))+$$
$$+\mu_i^{n+1/2}\frac{U_{ij+1}^{n+1/2}-U_{ij-1}^{n+1/2}}{2h_y}+\lambda_i^{n+1/2}U_{ij}^{n+1/2}+F_{ij}^{n+1/2}, \qquad (17)$$

At first the system dares (16), there is a decision on Intermediate step, then the system (17) also is the decision on unknown layer $n+1$. Matrixes of factors in systems (16), (17) have a three-diagonal appearance and dare Prorace method.

The analysis of the constructed scheme and the received results

In case of the obvious scheme, at $\sigma=1$ in expression (11) the differencing scheme approximates an initial problem with the first order on τ and second order on h_x and h_y, approximation error: $\varphi_{ij}^n=O(\tau+h_x^2+h_y^2)$. By means of the harmonious analysis the spectral sign of stability is deduced: $\frac{\tau}{h_x^2}+\frac{\tau}{h_y^2}\le\frac{1}{2}$.

In case of the implicit scheme, longitudinal-cross-section the differencing scheme approximates an initial problem with the second order on τ and the second order on h_x and h_y, approximation error: $\varphi_{ij}^n=O(\tau^2+h_x^2+h_y^2)$. But the implicit scheme certainly it is steady, i.e. stability of the decision does not depend on steps.

For the proof of stability of the decision constructed differencing problems (11) – (14) have been offered test functions and the numerical decision is constructed. At different steps differencing grids were different relative errors between the exact decision are received and the calculated. Dependence of an error on quantity of steps is revealed grids: the more the quantity of points of splitting, the is less received at calculations a relative error.

Conclusion

In work the two-dimensional inverse regional problem for the equation is solved heat conductivity. Using redefinition conditions, together with the problem decision defines unknown factors at $U_y(t,x,y)$ and $U(t,x,y)$ depending from t and x. In work the method of the decision based on the theory of differencing schemes is offered. The offered method is realized numerically. Experiments are made on various test functions and various steps of a grid. Also the constructed decision differencing problems is investigated on convergence and Stability.

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