APPLICATION OF MULTI STEP METHODS TO THE SOLVING SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

Inara Nasirova[1], Galina Mehdiyeva[2], Vagif Ibrahimov[3]

Baku State University, Baku, Azerbaijan
[1]*omega11@rambler.ru*, [2,3]*ibvag@yahoo.com*

When we speak on numerical solution of the initial value problem for higher order differential equations it is assumed that they may be reduced to the initial problem for a system of first order differential equations by means of change of variables and then using a wide store of numerical methods for solving first order ordinary differential equations to solve the initial value problem for received system. However, the methods specially constructed for solving higher order differential equations are more effective. Here, for illustration that we constructed one method and showed its efficiency.

Introduction.

We investigate numerical solution of the initial value problem for second order nonlinear ordinary differential equations in the following form:

$$y'' = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0. \tag{1}$$

Suppose that problem (1) has a unique continuous solution determined on the segment $[x_0, X]$. Finding its approximate values, by means of the constant step size $0 < h$, we partition the segment $[x_0, X]$ into N equal parts. We determine the partitioning points as $x_m = x_0 + mh$, and denote the approximate values of the solution of problem (1) at the point x_m by y_m ($m = 0,1,2,...,N$).

The scientists of the world study the solution of ordinary differential equations from Neuton's time. Many well known scientists beginning from Euler devoted their works to investigation of numerical solutions to initial value problem of the first order ordinary differential equations and applied the obtained result to the solution of problem (1) by means of the system of first order differential equations. While studying the orbit of celestial bodies, some scientists obtained problem (1) wherein the function $f(x, y, z)$ is independent of the argument z, i.e. $f(x, y, z) = F(x, y)$ and they constructed a method that was make effective than the known ones. The Stoermer's method is the more popular among them.

We can write Stoermer's generalized method in the following form [see (1)-(4)]

$$\sum_{i=0}^{k} \alpha'_i y_{n+1} = h^2 \sum_{i=0}^{k} \beta'_i F(x_{n+i}, y_{n+i}), \quad (n = 0,1,2,...) \tag{2}$$

where the coefficients $\alpha'_i, \beta'_i \quad (i = 0,1,2,...,k)$ are some real numbers, moreover $\alpha'_k \neq 0$, and the integer valued quantity k is called an order of difference method. Approximate values of the solution of the initial problem at the point $x_m \quad (m = 0,1,...,\mathrm{N})$ are denoted by y_m.

Notice that if we reduce problem (1) to the system of first order equations and get the following method

$$\sum_{i=0}^{k} \overline{\alpha} z_{n+i} = h \sum_{i=0}^{k} \overline{\beta}_i z'_{n+i} \tag{3}$$

for its solution, as a result we'll get a system of two finite-difference methods.

And of we compare the obtained one with Stermer's method, we see that the use of one finite-difference method is best than to use their systems. This advantage of Stermer's method shows itself more in using implicit schemes. However, method (2) also has some lacks that promoted to construct another method for solving special type ordinary differential equations (both hybrid and forward jumping types).

Here, for numerical solution of problem (1) we suggest a multi-step method with three derivatives that is researched in the following item.

1. Multi-step method with third derivative.

As is known, the finite-difference method (3) that is already classic, was investigated well by many authors (see [1]-[5]) as a numerical method for solving the initial problem for ordinary differential equations. One of the well investigated multi-step methods with constant coefficients is the following method

$$\sum_{i=0}^{k}\hat{\alpha}_i y_{n+i} = h\sum_{i=0}^{k}\hat{\beta}_i y'_{n+i} + h^2\sum_{i=0}^{k}\hat{\gamma}_i f\left(x_{n+i}, y_{n+i}, y'_{n+i}\right), \quad (n = 0,1,2,...). \tag{1.1}$$

Here, the coefficients $\hat{\alpha}_i, \hat{\beta}_i, \hat{\gamma}_i \quad (i = 0,1,2,...,k)$ are some real numbers $\hat{\alpha}_k \neq 0$, and y'_m is an approximate value of the first derivative of the solution of problem (1) at the point $x_m \ (m = 0,1,2,...)$.

It is easy to see that method (2) in obtained from (1.1) in particular for β_i=0 (i=0,1,2,…k). But peculiarities of these methods don't coincide completely, i.e. in some cases their peculiarities differ. For example, method (1.1) may be stable, but we can't say this about method (2).

As is known, stability of method (1.1) is a necessary and sufficient condition for its convergence. Generally speaking, the notion of stability of the method is determined by the coefficients of its linear part, in the present case, by the coefficients $\hat{\alpha}_i$ (i=0,1,2,…,k). Method (1.1) is said to be stable if the roots if its characteristical polynomial $p(\lambda) = \hat{\alpha}_k\lambda^k + ... + \hat{\alpha}_1\lambda + \hat{\alpha}_0$ lie interior to a unit circle with no multiple roots on the boundary. However, there is no method of type (2) whose characteristical polynomial has a non-multiple root $\lambda = 1$, since two-fold property of the root $\lambda = 1$ is a necessary condition of its convergence.

Thus, we showed that methods (2) and (1.1) in some cases have different peculiarities and therefore these methods are investigated separately.

In this report, to the numerical solution of problem (1) we suggest the following multi-step method with the third derivative

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i^{(1)} y'_{n+i} + h^2\sum_{i=0}^{k}\beta_i^{(2)} f_{n+i} + h^3\sum_{i=0}^{k}\beta_i^{(3)} g_{n+i} \qquad (n = 0,1,2,...) \tag{1.2}$$

here, the coefficients α_i, $\beta_i^j \ (i = 0,1,...,k;\ j = 1,2,3)$ are some real numbers, $\alpha_k \neq 0$, and

$$g(x, y, y') = f'_x(x, y, y') + f'_y(x, y, y')y' + f'_{y'}(x, y, y')\, f(x, y, y'),$$
$$f_m = f(x_m, y_m, y'_m), \quad g_m = (x_m, y_m, y'_m) \quad (m = 0,1,2...).$$

Obviously, $|\beta_k^{(3)}| + |\beta_{k-1}^{(3)}| + \ldots + |\beta_0^{(3)}| \neq 0$, otherwise we could get method (1.1). However, in this case we meet calculation of the function $g(x, y, y')$ whence it is seen that for determining g_m it is necessary to find the quantities y'_m even in the case when the function $f(x, y, y')$ is independent of y'. Thus, we obtain that while solving problem (1) by means of method (1.2), the values of the quantities y'_m should be calculated in parallels with finding the quantities y_m. Consequently, for solving problem (1) we apply the system composed of two multi-step methods. The method of type (1.1) may be used to calculate y'_m.

As it was noted above, stability of the multi-step method is a necessary and sufficient condition for its convergence. Therefore, stable methods are both of theoretical and practical interest. As is known (see[1]), if method (1.2) is stable, its power satisfies the condition $p \leq 3m + 4$ (see[8]) (k is an order of method (1.1) or (1.2)).

Usually, the integer-value quantity p is said to be a power of method (1.2) if for sufficiently smooth function it hold the following:

$$\sum_{i=0}^{k}\left(\alpha_i y(x+ih)-h\beta_i^{(1)}y'(x+ih)-h^2\beta_i^{(2)}y''(x+ih)-h^3\beta_i^{(3)}y'''(x+ih)\right)=O\left(h^{p+1}\right), h\to 0$$

Power of method (1.1) is determined in the same way.

We can prove that if methods (1.1) and (1.2) are stable, $\hat{\alpha}_k \neq 0$, $\alpha_k \neq 0$, are of power p_1 and p respectively, then the method composed of the methods of type (1.1) and (1.2)converges to the solution of problem (1) and convergence rate equals $p_1 + p$ for $p \le p_1 + 1$. Hence we get that under numerical solution of problem (1) by means of the method composed of the formulae of type (1.1) and (1.2), it is possible to select the method of type (1.2) with the best peculiarity. For example, for k>3, in place of the method of type (1.1) one can use stable implicit methods with maximal power P_{max}=2k+2, and in place of the method of type (1.2) on can use stable explicit methods with power P=3k; in this case, we can also select the methods with extended domain of stability.

For finding the values of $y'(x)$, we can use the following scheme. Integrating the differential equation on the segment $[x_0,x]$, we have

$$y'(x)=y'(x_0)+\int_{x_0}^{x} f\left(s,y(s),y'(s)\right)ds, \tag{1.3}$$

that is a Volterra type integral equation. The quadrate method is a traditional method for solving integral equations. However, to the numerical solution of equation (1.3) we can apply the multi-step method and get

$$\sum_{i=0}^{k}\alpha_i' y_{n+i}' = h\sum_{i=0}^{k}\beta_i' f\left(x_{n+i}, y_{n+i}, y_{n+i}'\right). \tag{1.4}$$

In order to construct the method having a higher accuracy than method (1.4), the functions $f(s,y(s),y'(s))$ are replaced by an interpolation polynomial, and as a result we get the following method

$$\sum_{i=0}^{k}\alpha_i y_{n+i}' = h\sum_{i=0}^{k}\beta_i f_{n+i} + h^2\sum_{i=0}^{k}\gamma_i g_{n+i}, \tag{1.5}$$

that coincides with method (1.1).

Now, consider construction of specific methods for solving problem (1). To this end, we determine the coefficients of these methods from the following homogeneous system of linear algebraic equations:

$$\sum_{i=0}^{k}\alpha_i = 0,\ \sum_{i=0}^{k} i\alpha_i = \sum_{i=0}^{k}\beta_i^{(1)},\ \sum_{i=0}^{k}\frac{i^2}{2!}\alpha_i = \sum_{i=0}^{k} i\beta_i^{(1)} + \sum_{i=0}^{k}\beta_i^2, \tag{1.6}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sum_{i=0}^{k}\frac{i^l}{l!}\alpha_i = \sum_{i=0}^{k}\frac{i^{l-1}}{(l-1)!}\beta_i^{(1)} + \sum_{i=o}^{k}\frac{i^{l-2}}{(l-2)!}\beta_i^{(2)} + \sum_{i=0}^{k}\frac{i^{l-3}}{(l-3)!}\beta_i^{(3)},\ l=3,4,\ldots p(0!=1)$$

At first we construct stable explicit methods of type (1.2) with maximal degree P_{max}=6. There are several such methods. Cite the following ones. Solving system (1.6) for k=2, we determine the coefficients of the multi-step method with the third derivative

$$y_{n+2} = y_{n+1} + h(15y_{n+1}' - 13y_n')/2 - h^2(31f_{n+1} + 29f_n)/10 + h^3(111g_{n+1} - 49g_n)/120 \tag{1.7}$$

$$\begin{aligned} y_{n+2} &= (y_{n+1}+y_n)/2 + h(31y_{n+1}' - 25y_n')/4 - \\ &- h^2(63f_{n+1} + 57f_n)/20 + h^3(233g_{n+1} - 97g_n)/240 \end{aligned} \tag{1.8}$$

These methods are recurrent relations. Therefore, if y'_n and y'_{n+1},are known, it is easy to apply them to the solution of problem (1). For finding these values, we suggest the following method with power p=6 whose coefficients are determined from system (1.6) for k=2 $k = 2$ и $\beta_i^{(3)} = 0$ $(i = 0,1,2)$:

$$y'_{n+2} = y'_{n+1} + h(101f_{n+2} + 128f_{n+1} + 11f_n)/240 + h^2(-139g_{n+2} + 40g_{n+1} + 3g_n)/240 \qquad (1.9)$$

and also the following implicit method with the degree p=5

$$y'_{n+2} = (y'_{n+1} + y'_n)/2 + h(45f_{n+2} + 64f_{n+1} + 11f_n)/80 - h^2(4g_{n+2} + 3g_{n+1})/40 \qquad (1.10)$$

Usually, implicit methods are more precise than the explicit ones. But when we use it we get nonlinear equations and it doesn't always turn out well to find their solution. Recently, in such cases it is suggested to use predictor-corrector type methods and follow them.

As a predictor method we suggest the following stable method with the degree p=4:

$$y'_{n+2} = y'_{n+1} + h(-1f_{n+1} + 3f_n)/2 + h^2(17g_{n+1} + 7g_n)/12$$

In place of the correction method we can use formula (1.10) or (1.9).

References

1. G. Dahlquist Convergence and stability in the numerical integration of ordinary differential equations. Math. Scand. 1956, №4, pp. 33-53.
2. N.S. Bakhvalov. Numerical methods. (In Russian) Nauka, 1973, 631 p.
3. E. Hairer, S.P. Norsett, G. Wanner. Solving ordinary differential equations. (In Russian) M., Mir, 1990, 512 p.
4. A.Jr. Huta. An a priori bound of the round-off error in the integration by multistep difference method for the differential equation $y^{(s)} = f(x, y)$. Acta F.R.N. Univ.Comen.Math., 1979, No.34, pp. 51-56.
5. V.S. Ryabenkiy, A.F. Filipov. On the stability of difference equations. (In Russian) M., 1956, 171 p.
6. R.J. Lambert Computational methods in ordinary differential equations. Wiley, 1973.
7. L. Lapidus, J.H. Seinfeld Numerical solution of ordinary differential equations. New York, London, 1971, 300 p.
8. V.R. Ibrahimov. On the maximum degree of the *k*-step Obrechkoff's method. Bulletin of Iranian Mathematical Society, Vol. 20, No. 1, pp. 1-28 (2002).