DECISION MAKING: THE MATHEMATICAL GROUND OF TRANSITIVITY FORMULA

Vladimir Zhukovin

Institute of Cybernetics, Tbilisi, Georgia
vovazhukovin@rambler.ru

Any decision making procedure (model) begins from collection of initial information that is received by pair wise comparison of decision from *X*.

Let $X = \{X_i\}^{n}{}_{i=1}$ be the set of competitive decisions; *P=X x X* be Cartesian product of *X*;

A numerical skew – symmetrical function determinated on *P-Z* (x_i, x_k) is called Superiority Degree (SD). That is: $Z(x_i, x_k) = -Z(x_i, x_k)$

This conception of SD on one decision over another has been introduced at first in [1] in context of group (social) decision of the modern Decision making theory. The theoretical-mathematical construction of SD has been developed in [2].

The problem is to form a linear order in X to found the most acceptable decision. Naturally a linear order should be coordinated with initial information.

In this report we will ground the condition of transitivity given by formula:

$$\text{(TF)}\ Z(x_i, x_k) + Z(x_k, x_e) = Z(x_i, x_e) \tag{1}$$

1. Let L be a linear order on *X*

This is a binary, connected (initial information is complete) transitive **Preference Relations**:

JF $(x_i, x_k) \in L$ then decision x_i is not worse than decision x_k.

2. **Utility function** is a discrete numerical function on *X*:

$U = \{u_1, u_2, ..., u_i, ..., u_n\}$,

Where

$$U_i = U(x_i) \text{ and } (x_i, x_k) \in h \leftrightarrow u_i \geq u_k \tag{2}$$

(≤ is natural order on the memorial set)

3. We will denote the function V on P:

$$V(x_i, x_k) = u_i - u_k \tag{3}$$

Then

$$(x_i, x_k) \in L \leftrightarrow u_i \geq u_k \leftrightarrow V(x_i, x_k) = 0 \tag{4}$$

Now let us present the transitivity property of these three structures:

$$\begin{aligned} &1. (x_i, x_k) \in L \,\&\, (x_k, x_s) \in L \Rightarrow (x_i, x_s) \in L \\ &2. u_i \geq u_k \,\&\, u_k \geq u_s \Rightarrow u_i \geq u_s \\ &3. V(x_i, x_k) \geq 0 \,\&\, V(x_k, x_s) \geq 0 \Rightarrow V(x_i, x_s) \geq 0 \end{aligned} \tag{5}$$

The properties of the function $V(x_i, x_k)$

1. $V(x_i, x_k)$ is SD because it is skew-symmetric [1]

$$V(x_i, x_k) = -V(x_k, x_i) \tag{6}$$

2. The next formula holds:

$$V(x_i, x_s) - V(x_k, x_s) = V(x_i, x_k) \tag{7}$$

3. Combining (6) u (7) we get

$$V(x_i, x_s) + V(x_s, x_k) = V(x_i, x_k) \tag{8}$$

Th.1.[2]

a) **If the notation (3) holds for SD, then transitivity formula (TF) is correct for it**
b) **If TF is correct for any SD in the form of (1), then potential notation holds in the form of (3) for it**

Now we return to beginning of decision making procedure (model). Let us receive the complete initial information in the form of set of SD, defined on the P.

$$Z = \{zj(x_i, x_k)\}_j \tag{9}$$

If formula 8 is correct for this set, then the potential notation holds:

$$z(x_i, x_k) = w(x_i) - w(x_k) \tag{10}$$

The formula of transitivity in the form of (7) will be used for definition of the potentials entering in formula (10)

$$z(x_i, x_s) - z(x_k, x_s) = z(x_i, x_k) \tag{11}$$

The next formula is correct since the formula (11) holds V X_s £ X

$$\sum_{xs \in X} z(x_i, x_s) - \sum_{xs \in X} 2z(x_k, x_s) = h \times 2(x_i, x_k) \tag{12}$$

The next expression for evaluation of the potentials in formula (10) will be obtained by comparison formula (12) and (10)

$$w_i = w(x_i) = \frac{1}{n} \sum_{x_r \in X} z(x_i, x_r) \tag{13}$$

Let us compare formulae (3) and (10)

$$x(x_i, x_k) = z(x_i, x_k) \ \forall i, k = \overline{1, n} \tag{14}$$

Then next relation holds:

$$u(x_i) = w(x_i) + d_1$$

where d is arbitrary constant

The Transform (14) is allowed for order scales

Thus we have utility function W coordinated with U and hence with linear order L

$$U \leftrightarrow W = \{w_i\} \ i = \overline{1, n} \tag{15}$$

Thus transitivity formula (1) now is proved.

Integral total superiority Degree (ISD).

In practical situation of decision making alternative version many happen when the initial information doesn't agree with transitivity formula.

In this care we present to form integral total superiority degree– ISD. In addition, the competing decisions x_i and is equal with regard to impact of ACL alternative decisions $x_s \in X$ including x_i and x_k.

This fact we may be presented by the next formula:

$$\tilde{Z}(x_i, x_k) = \frac{1}{n} \sum_{x_s \in X} z(x_i, x_s) - \frac{1}{n} \sum_{x_s \in X} z(x_k, x_s) \tag{16}$$

This is a numerical function defined on the decisions pairs set P. It is skew–symmetric and hence it is SD. In addition transitivity formula (1) ALWAYS is realized for it. This fact may verified substitution (16) in (1).

Then we may write the next formulae according to with theorem 1:

1. $\tilde{Z}(x_i, x_k) = C_i(x_i) - C_i(x_k)$ (17)

2. $C_i(x_i) = \frac{1}{n} \sum_{x_s \in X} \tilde{z}(x_i, x_s)$ (18)

Let us substitute (16) in (18) and simplify the obtained expression. Then the formula is correct:

$$C_i(x_i) = \frac{1}{n} \sum_{x_r \in X} z(x_i, x_r) \tag{19}$$

Only it is necessary to consider that:

$$\sum_{x_s \in X} \sum_{x_r \in X} z(x_s, x_r) = 0 \tag{20}$$

Thus, the formulae (13) and (19) coincide, and we have next formula:

$$w(x_i) = C_1(x_i) + d \tag{21}$$

where d is an arbitrary constant.

These formulae allow to form a linear order on X using initial information (9) coordinated with this information. The formula (19) may be obtained by a comparison of (17) and (16).

References

1. V.E. Zhukovin. K probleme gruppovogo vibora. The reports of Academy of Science of Georgia, Tbilisi 99, N2, 1980/
2. V. Zhukovin, Z. Alimbarashvili. Decision Making, Superiority Degree, N1, 2000.
3. Proceedings of the Institute of Cibernetics. VOL1, N 1, 2000.