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ESTIMATION OF NET PREMIUMS IN COLLECTIVE LIFE INSURANCE[1]

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Nonparametric estimates of net premiums in the collective models of insurance are proposed. The asymptotic normality and the mean square convergence of the proposed estimates are proved. The main parts of asymptotic mean square errors of net premiums estimates are found. Simulation results show that the nonparametric estimates are just as good in practice.

1. Collective Insurance. The concept of status is the useful abstraction in the collective life insurance according to [1]. Let us consider m members of the ages $x_1, x_2, ..., x_m$ those who so desire to buy insurance policy. Let us denote the future lifetime of k-th individual by $T(x_k) = X - x_k$. Let us put in a correspondence a status U with its future lifetime $T(U)$ and a set of numbers $T(x_1), T(x_2), ..., T(x_m)$. Joint-life status and last-survivor status are the most widespread.

Joint-life status is denoted by $U := x_1 : x_2 : ... : x_m$ and is considered as failed upon the first death, i.e., $T(U) = \min(T(x_1), T(x_2), ..., T(x_m))$. It is evident that probability $\mathbf{P}\{T(U) > t\} = = \mathbf{P}\{T(x_1) > t, T(x_2) > t, ..., T(x_m) > t\}$, so when the deaths are independent we have $\mathbf{P}\{T(U) > t\} = \prod_{i=1}^{m} \mathbf{P}\{T(x_i) > t\}$.

Last-survivor status is denoted by $U := \overline{x_1 : x_2 : ... : x_m}$ and fails upon the last death, and exists as long as at least one member of a group is alive, i.e.,

$$T(U) = \max(T(x_1), T(x_2), ..., T(x_m)).$$

Similarly,

$$\mathbf{P}\{T(U) \le t\} = \mathbf{P}\{\max(T(x_1), T(x_2), ..., T(x_m)) \le t\} =$$
$$= \mathbf{P}\{T(x_1) \le t, T(x_2) \le t, ..., T(x_m) \le t\},$$

and in the case of independent deaths we have $\mathbf{P}\{\max(T(U) \le t\} = \prod_{i=1}^{m} \mathbf{P}\{T(x_i) \le t\}$.

2. Functionals of Net Premiums. Consider the last-survivor status of two members $U := \overline{x_1 : x_2}$ with $T(U) = \max(T(x_1), T(x_2))$. Analogously to the case of individual whole life insurance [2] the net premium can be written in the form

$$\bar{A}_{\overline{x_1:x_2}} = \int_0^\infty \exp(-\delta t)\; f_{\overline{x_1:x_2}}(t)\, dt,$$

where δ is the interest rate and the probability density function of the last-survivor status is

$$f_{\overline{x_1:x_2}}(t) = \left(\mathbf{P}\{\max(T(x_1), T(x_2) \le t\}\right)_t^{/} =$$
$$= f_{x_1}(t) F_{x_2}(t) + f_{x_2}(t) F_{x_1}(t) ,$$

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where $F_x(t)$ and $f_x(t)$ are the distribution function and the curve of death for random variable $T(x)$ accordingly. Thus,

$$\bar{A}_{\overline{x_1:x_2}} = \int_0^{\infty} \exp(-\delta\, t)\,[\, f_{x_1}(t) F_{x_2}(t) + \ f_{x_2}(t) F_{x_1}(t)]\, dt. \qquad (1)$$

Using (1) we can see that the last-survivor status net premium $\bar{A}_{\overline{x_1:x_2}}$ less than the sum of individual net premiums $\bar{A}_{x_1}$ and $\bar{A}_{x_2}$ in view of $F_{x_1}(t) \le 1$, $F_{x_2}(t) \le 1$. Note that we can use formula (1) in parametric estimation.

For example, the de Moivre model has the following characteristics:

$$f_x(t) = \frac{I_t(0, \omega - x)}{\omega - x}, \ F_x(t) = I_t(\omega - x, \infty) + \frac{t\, I_t(0, \omega - x)}{\omega - x},$$

where $\quad I_t(a,b) = \begin{cases} 1, & t \in (a,b) \\ 0, & t \notin (a,b) \end{cases}$, and functional (1) is written as

$$\bar{A}_{\overline{x_1:x_2}} = \frac{\exp[-\delta(\omega - x_1)] - \exp[-\delta \max(\omega - x_1, \omega - x_2)]}{\delta(\omega - x_1)} +$$
$$+ \frac{\exp[-\delta(\omega - x_2)] - \exp[-\delta \max(\omega - x_1, \omega - x_2)]}{\delta(\omega - x_2)} +$$
$$+ \frac{2 - [2\delta \min(\omega - x_1, \omega - x_2) + 2] e^{-\delta \min(\omega - x_1, \omega - x_2)}}{\delta(\omega - x_1)(\omega - x_2)},$$

where ω is the limiting age. If the parameter ω is unknown we can use the following estimate: $\hat{\omega} = \max(X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n)$, where $(X_1, Y_1), (X_2, Y_2), ..., (X_n, Y_n)$ is the sample of moments of deathes of pairs of individuals.

Functionals (1) for the Gompertz, Makeham and Weibull models are estimated by computed methods, and unknown parameters are found by method of moments. The tables of net premiums for different ages are designed. It is important note that Makeham model characterizes the mortality process the most adequately.

In general the net premium functional of the last-survivor status of two members is written as

$$\bar{A}_{\overline{x_1 x_2}} = \frac{1}{\bar{S}(x_1, x_2)} \int_0^{\infty} \exp(-\delta t)\; d\,\mathbf{P}\{\max(X - x_1, Y - x_2) \le t\} = \frac{\bar{\Phi}(x_1, x_2, \delta)}{\bar{S}(x_1, x_2)},$$

where $\bar{S}(x_1, x_2) = \mathbf{P}\{\max(X - x_1, Y - x_2) > 0\}$ is the survival function of the last-survivor status.

3. The Estimates of Net Premiums and Their Properties. Let two-dimensional sample of independent and identically distributed random variables be $(X_1, Y_1), (X_2, Y_2), ..., (X_n, Y_n)$. We estimate the distribution function $\mathbf{P}\{\max(X - x_1, Y - x_2) \le t\}$ and the survival function $\bar{S}(x_1, x_2)$ by the unbiased nonparametric estimates $\frac{1}{n}\sum_{i=1}^{n} I(\max(X_i - x_1, Y_i - x_2) \le t)$ and $\frac{1}{n}\sum_{i=1}^{n} I(\max(X_i - x_1, Y_i - x_2) > 0)$, where $I(A)$ is an indicator of the set A. Following reasonings for a single client model (see [3]), we obtain:

$$\hat{\bar{A}}_{\overline{x_1:x_2}} = \frac{1}{\bar{S}_n(x_1,x_2)}\int_0^\infty \exp(-\delta t)\ d\left(\frac{1}{n}\sum_{i=1}^n I(\max(X_i - x_1, Y_i - x_2 \le t)\right) =$$

$$= \frac{1}{\bar{S}_n(x_1,x_2)}\int_0^\infty \exp(-\delta t)\ \bar{\delta}(t - \max(X_i - x_1, Y_i - x_2))dt =$$

$$= \frac{\sum_{i=1}^n \exp(-\delta \max(X_i - x_1, Y_i - x_2))}{n\bar{S}_n(x_1,x_2)}\ I(\max(X_i - x_1, Y_i - x_2) > 0) = \frac{\bar{\Phi}_n(x_1,x_2,\delta)}{\bar{S}_n(x_1,x_2)}. \qquad (2)$$

Here $\bar{\delta}(x)$ is the known Dirac delta-function.

Theorem 1. Let $\bar{S}(x_1,x_2)$ be the survival function of the last-survivor status and $\bar{S}(x_1,x_2) \ne 0$. Then the estimate of net premium $\hat{\bar{A}}_{\overline{x_1:x_2}}$ is asymptotic unbiased, i.e.,

$$| E(\hat{\bar{A}}_{\overline{x_1:x_2}}) - \bar{A}_{\overline{x_1:x_2}} | = O\left(n^{-1}\right),$$

and mean square error is equal to

$$u^2(\hat{\bar{A}}_{\overline{x_1:x_2}}) = \frac{\bar{\Phi}(x_1,x_2,2\delta)\bar{S}(x_1,x_2) - \bar{\Phi}^2(x_1,x_2,\delta)}{n\bar{S}^3(x_1,x_2)} + O\left(n^{-3/2}\right).$$

Proof of Theorem 1 is based on using the results of the chapter 9 [4].

To find the limiting distribution of the net premium estimate $\hat{\bar{A}}_{\overline{x_1:x_2}}$ we use the theorem 1.7.1 [4]. Let us denote by $\Rightarrow N_1(a,b)$ the symbol of weak convergence of sequence of random variables to a normal random variable with mean a and variance b.

Theorem 2. If the survival function of the last-survivor status $\bar{S}(x_1,x_2) \ne 0$, then $\sqrt{n}[\hat{\bar{A}}_{\overline{x_1:x_2}} - \bar{A}_{\overline{x_1:x_2}}] \Rightarrow N_1\{0, Q_{\overline{x_1:x_2}}\}$, where $Q_{\overline{x_1:x_2}} = \frac{\bar{\Phi}(x_1,x_2,2\delta)\bar{S}(x_1,x_2) - \bar{\Phi}^2(x_1,x_2,\delta)}{n\bar{S}^3(x_1,x_2)}$.

4. Modification of Estimate (2). Note that estimate $\hat{\bar{A}}_{\overline{x_1:x_2}}$ has the shortcoming, because sometimes $\bar{S}_n(x) = 0$. This shortcoming is overcomed by use of truncated modification $\hat{\bar{A}}_{\overline{x_1:x_2}}$ and piecewise-smoothed approximation $\tilde{\bar{A}}_{\overline{x_1:x_2}}$ [5,6]:

$$\tilde{\bar{A}}_{\overline{x_1:x_2}} = \begin{cases} \hat{\bar{A}}_{\overline{x_1:x_2}}, & if\ \hat{\bar{A}}_{\overline{x_1:x_2}} \le Cn^\gamma \\ Cn^\gamma, & otherwise \end{cases}, \quad C, \gamma > 0, \qquad (3)$$

$$\tilde{\bar{A}}_{\overline{x_1:x_2}} = \frac{\hat{\bar{A}}_{\overline{x_1:x_2}}}{(1 + \delta_n \hat{\bar{A}}^\tau_{\overline{x_1:x_2}})^\rho}, \quad \tau > 0,\ \rho > 0,\ \rho\tau \ge 1,\ \delta_n = O(n^{-1}). \qquad (4)$$

The analogues of Theorem 1 and Theorem 2 for estimates (3) and (4) are proved too.

5. The Case of m Clients. In the case of m clients the functional of net premium can be written as

$$\bar{A}_{\overline{x_1:\ldots:x_m}} = \frac{1}{\bar{S}(x_1,\ldots,x_m)}\int_0^\infty \exp(-\delta t)\ d\,\mathbf{P}\{\max(X_1 - x_1, \ldots, X_m - x_m) \le t\}.$$

Let $(X_{11},...X_{1m}),...,(X_{n1},...,X_{nm})$ be the m-dimensional sample. Then in the manner similar to (2), the estimate of the net premium is

$$\hat{A}_{\overline{x_1:...:x_m}} = \frac{\sum_{i=1}^{n} \exp(-\delta \max(X_{i1}-x_1,...X_{im}-x_m))}{n\bar{S}_n(x_1,...,x_m)} \times$$

$$\times I(\max(X_{i1}-x_1,...,X_{im}-x_m) \le t) .$$

This estimate is asymptotically unbiased, i.e., $|E(\hat{A}_{\overline{x_1:...:x_m}}) - \bar{A}_{\overline{x_1:...:x_m}}| = O\left(n^{-1}\right)$, its mean square error is

$$u^2(\hat{A}_{\overline{x_1:...:x_m}}) = \frac{\bar{\Phi}(x_1,\ldots,x_m,2\delta)\,\bar{S}(x_1,x_2) - \bar{\Phi}^2(x_1,\ldots,x_m,\delta)}{n\bar{S}^3(x_1,\ldots,x_m)} + O\left(n^{-3/2}\right).$$

It is easily to obtain results that are analogous to the above-mentioned results for the net premiums estimates for models of p-year term life insurance, p-year pure endowment and r-year deferred insurance.

6. Simulation Results. The nonparametric estimates show adaptability if the distribution is changed and exceed parametric estimates, oriented on the best result only for its own distributions. Often the mean square error of nonparametric estimates are less than the mean square error of nonparametric estimates in 2-3 times. The main modeling results obtained using data from Makeham and de Moivre distributions.

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