EXTREME PROBLEM FOR THREE-DIMENSIONAL DIFFERENTIAL INCLUSIONS

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Extreme problems for differential inclusions allow to cover many of the considered problems of optimal control. Extreme problems for various types of multidimensional differential inclusions are investigated by the author.

In the paper (see [1]) the problems of a minimum for extreme problems of three-dimensional differential inclusions are considered. The work consists of four parts. In p.1 are investigated continuous dependence of decisions of three-dimensional differential inclusions on the right part and on boundary conditions (see. [2]) are investigated. In p.2 variations problems in which extreme problems results for three-dimensional differential inclusions are considered. In p.3 extreme problems for three-dimensional differential inclusions are considered, necessary conditions of the first order are obtained. In p.4 the necessary conditions of the second order of problems a minimum for three-dimensional differential inclusions are obtained.

Let $f:[0,1]^3\times R^{2(k_1+k_2+k_3)}\to\overline{R}=R\bigcup\{+\infty\}$, $\varphi_1:[0,1]^2\times R^{2k_1}\to\overline{R}$, $\varphi_2:[0,1]^2\times R^{2k_2}\to\overline{R}$, $\varphi_3:[0,1]^2\times R^{2k_2}\to\overline{R}$ be normal integrands, $a_i:[0,1]^3\times R^{k_1+k_2+k_3}\to 2^{R^{k_i}}$, where $i=1,2,3$; k_1,k_2,k_3 are natural numbers, $a_i(x,y,z,\omega)$ are compact at any (x,y,z,ω), $M_i:[0,1]^2\to 2^{R^{k_i}}$, $i=1,2,3,$ are measurable; in addition $M_1(y,z)$, $M_2(x,z)$ and $M_3(x,y)$ are non empty at $x,y,z\in[0,1]$. We put
$V_x^p=\{u\in L_p^{k_1}[0,1]^3:u_x\in L_p^{k_1}[0,1]^3\}$, $V_y^p=\{\upsilon\in L_p^{k_2}[0,1]^3:\upsilon_y\in L_p^{k_2}[0,1]^3\}$,
$V_z^p=\{w\in L_p^{k_3}[0,1]^3:w_z\in L_p^{k_3}[0,1]^3\}$,
where $1\le p<+\infty$. If $(u,\upsilon,w)\in V_x^p\times V_y^p\times V_z^p$, then we denote

$$U(x,y,z)=(u(x,y,z),\upsilon(x,y,z),w(x,y,z)),$$
$$\widetilde{\nabla}U(x,y,z)=(u_x(x,y,z),\upsilon_y(x,y,z),w_z(x,y,z)).$$

The function $(u,\upsilon,w)\in V_x^p\times V_y^p\times V_z^p$ satisfying inclusions

$$u_x(x,y,z)\in a_1(x,y,z,U(x,y,z)),\upsilon_y(x,y,z)\in a_2(x,y,z,U(x,y,z)),$$
$$w_z(x,y,z)\in a_3(x,y,z,U(x,y,z)),$$
$$u(0,y,z)\in M_1(y,z),\ \upsilon(x,0,z)\in M_2(x,z)\ ,w(x,y,0)\in M_3(x,y)\ at\ \ x,y,z\in[0,1] \qquad (1)$$

is called the solution of inclusion (1).

The solution of inclusion (1) minimizing of function

$$J_p(u,v,w)=\int_0^1\int_0^1\int_0^1 f(x,y,z,U(x,y,z),\widetilde{\nabla}U(x,y,z))dxdydz+$$
$$+\int_0^1\int_0^1 \varphi_1(y,z,u(0,y,z),u(1,y,z))dydz+\int_0^1\int_0^1 \varphi_2(x,z,\upsilon(x,0,z),\upsilon(x,1,z))dxdz+ \qquad (2)$$
$$+\int_0^1\int_0^1 \varphi_3(x,y,w(x,y,0),w(x,y,1))dxdy$$

among all solution of inclusion (1) is called an optimal. It is required to find necessary extremum conditions of the solution of a problem (1), (2).

Let

$\psi_1(x,y,z,\omega,u)=\inf\{|u_1-u|:u_1\in a_1(x,y,z,\omega)\}$,

$q_1(y,z,u)=\inf\{|u_1-u|:u_1\in M_1(y,z)\}$,

$\psi_2(x,y,z,\omega,\upsilon)=\inf\{|\upsilon_1-\upsilon|:\upsilon_1\in a_2(x,y,z,\omega)\}$,

$q_2(y,z,\upsilon)=\inf\{|\upsilon_1-\upsilon|:\upsilon_1\in M_2(x,z)\}$,

$\psi_1(x,y,z,\omega,w)=\inf\{|w_1-w|:w_1\in a_3(x,y,z,\omega)\}$,

$q_3(x,y,w)=\inf\{|w_1-w|:w_1\in M_3(x.y)\}$.

$$F_p(u,\upsilon,w)=\{\int_0^1\int_0^1\int_0^1(\psi_1(x,y,z,U(x,y,z),\widetilde{\nabla}U(x,y,z))+\psi_2(x,y,z,U(x,y,z),\widetilde{\nabla}U(x,y,z))+$$
$$+\psi_3(x,y,z,U(x,y,z),\widetilde{\nabla}U(x,y,z)))^p dxdydz\}^{\frac{1}{p}}+(\int_0^1\int_0^1 q_1^p(y,z,u(0,y,z))dydz)^{\frac{1}{p}}+$$
$$+(\int_0^1\int_0^1 q_2^p((x,z,\upsilon(x,0,z))dxdz)^{\frac{1}{p}}+(\int_0^1\int_0^1 q_3^p((x,y,w(x,y,0))dxdy)^{\frac{1}{p}},$$
$$I_m(u,\upsilon,w)=J_p(u,\upsilon,w)+m\cdot F_p(u,\upsilon,w).$$

Theorem 1. Let mappings $(x,y,z)\to a_i(x,y,z,\omega)$, $i=1,2,3$, $(y,z)\to M_1(y,z)$, $(x,z)\to M_2(x,z)$ and $(x,y)\to M_3(x,y)$ are measurable, $a_i(x,y,z,\omega)$, $M_1(y,z)$, $M_2(x,z)$ and $M_3(x,y)$ are non empty, compact at $x,y,z\in\in[0,1],\omega\in R^{k_1+k_2+k_3}$ and $\rho_x\big(a_i(x,y,z,\omega),a_i(x,y,z,\omega_1)\big)\le M|\omega_1-\omega|$ holds at $\omega,\omega_1\in R^{k_1+k_2+k_3}$, $i=1,2,3$. Besides let there exists functions $k(\cdot)\in L_q([0,1]^3)$, $k_1(\cdot),\,k_2(\cdot),\,k_3(\cdot)\in L_q[0,1]^2$, where $\dfrac{1}{p}+\dfrac{1}{q}=1$, $p\in[1,+\infty)$, such, that

$$|f(x,y,z,\omega)-f(x,y,z,\omega_1)|\le k(x,y,z)|\omega-\omega_1| \quad \text{for } \omega,\omega_1\in R^{2(k_1+k_2+k_3)},$$
$$|\varphi_1(y,z,u)-\varphi_1(y,z,u_1)|\le k_1(y,z)|u-u_1| \quad \text{for } u,\,u_1\in R^{2k_1},$$
$$|\varphi_2(x,z,\upsilon)-\varphi_2(x,z,\upsilon_1)|\le k_2(x,z)|\upsilon-\upsilon_1| \quad \text{for } \upsilon,\,\upsilon_1\in R^{2k_2},$$
$$|\varphi_3(x,y,\varpi)-\varphi_2(x,z,\varpi_1)|\le k_2(x,y)|\varpi-\varpi_1| \quad \text{for } \varpi,\varpi_1\in R^{2k_3},$$

and let mappings $(x,y,z)\to f(x,y,z,\omega)$, $(y,z)\to\varphi_1(y,z,u)$, $(x,z)\to\varphi_2(x,z,\upsilon)$, $(x,y)\to\varphi_3(x,y,\varpi)$ are measurable. Then, if $(\overline{u},\overline{\upsilon},\overline{w})\in V_x^p\times V_y^p\times V_z^p$ is the solution of a problem (1), (2), then there a number $m_0 > 0$ exists such, that

$$I_{m_0}^{\{\beta\}-}((\bar{u},\bar{\upsilon},\bar{w});(u,\upsilon,w)) = \varliminf_{\lambda\downarrow 0}\frac{I_{m_0}((\bar{u},\bar{\upsilon},\bar{w})+\lambda(u,\upsilon,w)) - I_{m_0}(\bar{u},\bar{\upsilon},\bar{w})}{\lambda^\beta} \ge 0$$

for $(u,\upsilon,w)\in V_x^p\times V_y^p\times V_z^p$, $\beta>0$.

Remark 1. Note, that if $k(\cdot)\in L_\infty([0,1]^3)$, $k_1(\cdot),\, k_2(\cdot),\, k_3(\cdot)\in L_\infty[0,1]^2$ then similarly theorem 1 it is proved, that exists $m>0$ such, that $(\bar{u},\bar{\upsilon},\overline{\mathrm{w}})$ also minimizes functional $I_m(u,\upsilon,w) = J_p(u,\upsilon,w) + m\cdot F_1(u,\upsilon,w)$ in space $V_x^p\times V_y^p\times V_z^p$.

Corollary 1. Let the conditions of theorem 1 are satisfied, in addition functions $\omega\to f(x,y,z,\omega)$, $u\to\varphi_1(y,z,u)$, $\upsilon\to\varphi_2(x,z,\upsilon)$, $\varpi\to\varphi_3(x,y\varpi)$ are convex, mappings $\omega\to a_i(x,y,z,\omega)$, $i=1,2,3$, are convex, sets $M_1(y,z)$, $M_2(x,z)$ and $M_3(x,y)$ are convex at $(x,y,z)\in[0,1]^3$, $k(\cdot)\in L_\infty([0,1]^3)$, $k_1(\cdot),\, k_2(\cdot),\, k_3(\cdot)\in L_\infty[0,1]^2$. Then for that $(\bar{u},\bar{\upsilon},\overline{\mathrm{w}})$ was a minimum point of function $J(u,\upsilon,\mathrm{w})$ in space $V_x^p\times V_y^p\times V_z^p$, is necessary and sufficiently to find functions $(\bar{u}^*,\bar{\upsilon}^*,\bar{w}^*)\in L_q^{k_1+k_2+k_3}([0,1]^3)$, where $\bar{u}_x^*\in L_q^{k_1}([0,1]^3)$, $\bar{\upsilon}_y^*\in L_q^{k_2}([0,1]^3)$, $\bar{w}_z^*\in L_q^{k_3}([0,1]^3)$ and the number $m>0$ such, that

1) $(\bar{u}_x^*(x,y,z),\bar{\upsilon}_y^*(x,y,z),\bar{w}_z^*(x,y,z),\bar{u}^*(x,y,z),\bar{\upsilon}^*(x,y,z),\bar{w}^*(x,y,z))\in$
$\in\partial\bar{f}(x,y,z,\bar{u}(x,y,z),\bar{\upsilon}(x,y,z),\bar{w}(x,y,z),\bar{u}_x(x,y,z),\bar{\upsilon}_y(x,y,z),\bar{w}_z(x,y,z))$,

2) $(\bar{u}^*(0,y,z),-\bar{u}^*(1,y,z)))\in\partial(\varphi_1(y,z,\bar{u}(0,y,z),\bar{u}(1,y,z)) + m\mathrm{q}_1(y,z,\bar{u}(0,y,z)),$

3) $(\bar{\upsilon}^*(x,0,z),-\bar{\upsilon}^*(x,1,z)))\in\partial(\varphi_2(x,z,\bar{\upsilon}(x,0,z),\bar{\upsilon}(x,1,z)) + m\mathrm{q}_2(x,z,\bar{\upsilon}(x,0,z)),$

4) $(\bar{w}^*(x,y,0),-\bar{w}^*(x,y,1)))\in\partial(\varphi_3(x,y,\bar{w}(x,y,0),\bar{w}(x,y,0)) + m\mathrm{q}_3(x,y,\bar{w}(x,y,0)),$

where $\bar{f} = f + m(\psi_1+\psi_2+\psi_3)$.

Theorem 2. Let $(\bar{u},\bar{\upsilon},\bar{w})\in V_x^p\times V_y^p\times V_z^p$ $(1\le p<+\infty\}$ is the solution of a problem (1), (2) and the conditions of theorem 1 are satisfied. Besides $k(\cdot)\in L_\infty([0,1]^3)$, $k_1(\cdot),\, k_2(\cdot),\, k_3(\cdot)\in L_\infty[0,1]^2$. Then there exists number $m>0$ and the functions $(\bar{u}^*,\bar{\upsilon}^*,\bar{w}^*)\in L_q^{k_1+k_2+k_3}([0,1]^3)$, where $\bar{u}_x^*\in L_q^{k_1}([0,1]^3)$, $\bar{\upsilon}_y^*\in L_q^{k_2}([0,1]^3)$, $\bar{w}_z^*\in L_q^{k_3}([0,1]^3)$, such, that

1) $(\bar{u}_x^*(x,y,z),\bar{\upsilon}_y^*(x,y,z),\bar{w}_z^*(x,y,z),\bar{u}^*(x,y,z),\bar{\upsilon}^*(x,y,z),\bar{w}^*(x,y,z))\in$
$\in\partial\bar{f}(x,y,z,\bar{u}(x,y,z),\bar{\upsilon}(x,y,z),\bar{w}(x,y,z),\bar{u}_x(x,y,z),\bar{\upsilon}_y(x,y,z),\bar{w}_z(x,y,z))$,

2) $(\bar{u}^*(0,y,z),\bar{u}^*(1,y,z)))\in\partial(\varphi_1(y,z,\bar{u}(0,y,z),\bar{u}(1,y,z)) + m\mathrm{q}_1(y,z,\bar{u}(0,y,z)),$

3) $(\bar{\upsilon}^*(x,0,z),\bar{\upsilon}^*(x,1,z)))\in\partial(\varphi_2(x,z,\bar{\upsilon}(x,0,z),\bar{\upsilon}(x,1,z)) + m\mathrm{q}_2(x,z,\bar{\upsilon}(x,0,z)),$

4) $(\bar{w}^*(x,y,0),\bar{w}^*(x,y,1)))\in\partial(\varphi_3(x,y,\bar{w}(x,y,0),\bar{w}(x,y,1)) + m\mathrm{q}_3(x,y,\bar{w}(x,y,0)),$

where $\bar{f} = f + m(\psi_1+\psi_2+\psi_3)$.

We denote $K_i(x,y,z,\omega,\vartheta) = \delta_{gr\,a_i(x,y,z,\omega)}(\omega,\vartheta) = \begin{cases} 0; & \vartheta\in a_i(x,y,z,\omega),\\ +\infty; & \vartheta\in a_i(x,y,z,\omega).\end{cases}$

Theorem 3. Let $(\bar{u},\bar{\upsilon},\bar{w})\in V_x^p\times V_y^p\times V_z^p$ $(1\le p<+\infty\}$ is the solution of a problem (1), (2) and the conditions of theorem 1 are satisfied. Besides $k(\cdot) \in L_\infty([0,1]^3)$, $k_1(\cdot), k_2(\cdot), k_3(\cdot) \in L_\infty[0,1]^2$. Then there exists number $m > 0$ and the functions $(\overline{u}^*, \overline{\upsilon}^*, \overline{w}^*) \in L_q^{k_1+k_2+k_3}([0,1]^3)$, where $\overline{u}_x^* \in L_q^{k_1}([0,1]^3)$, $\overline{\upsilon}_y^* \in L_q^{k_2}([0,1]^3)$, $\overline{w}_z^* \in L_q^{k_3}([0,1]^3)$, such, that

1) $(\overline{u}_x^*(x,y,z), \overline{\upsilon}_y^*(x,y,z), \overline{w}_z^*(x,y,z), \overline{u}^*(x,y,z), \overline{\upsilon}^*(x,y,z), \overline{w}^*(x,y,z)) \in$
$\in \partial \tilde{f}(x,y,z,\overline{u}(x,y,z), \overline{\upsilon}(x,y,z), \overline{w}(x,y,z), \overline{u}_x(x,y,z), \overline{\upsilon}_y(x,y,z), \overline{w}_z(x,y,z))$,

2) $(\overline{u}^*(0,y,z), -\overline{u}^*(1,y,z))) \in \partial\varphi_1(y,z,\overline{u}(0,y,z), \overline{u}(1,y,z)) + (N_{M_1(y,z)}(\overline{u}(0,y,z)),0)$,

3) $(\overline{\upsilon}^*(x,0,z), -\overline{\upsilon}^*(x,1,z))) \in \partial\varphi_2(x,z,\overline{\upsilon}(x,0,z), \overline{\upsilon}(x,1,z)) + N_{M_2(x,z)}(\overline{\upsilon}(x,0,z)),0)$,

4) $(\overline{w}^*(x,y,0), -\overline{w}^*(x,y,1))) \in \partial\varphi_3(x,y,\overline{w}(x,y,0), \overline{w}(x,y,1)) + N_{M_3(x,y)}(\overline{w}(x,y,0)),0)$,

where $\tilde{f} = f + (K_1 + K_2 + K_3)$.

References

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