# ON THE OPTIMIZATION OF INITIAL DATA FOR DELAY DIFFERENTIAL EQUATIONS

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**Abstract.** In this work, the necessary conditions of optimality for initial data are given for delay differential equations with discontinuous and continuous initial conditions. The discontinuous initial condition means that, in general, the values of the initial function and trajectory do not coincide at initial instant of time. The continuous initial condition means that the values of the initial function and trajectory always coincide at initial instant of time. Under initial data we imply the collection of initial moment and delay parameter, initial vector and function.

## 1. Equation with the discontinuous initial condition

Let $R^n$ be the $n$-dimensional vector space of points $x=(x^1,...,x^n)^T$, where $T$ means transpose; let $s_0<s_1<t_1$, $0<\tau_1<\tau_2$ be given numbers with $t_1-s_1>\tau_2$; suppose that $O\subset R_x^n$ is an open set , $K\subset O$ is a compact set, $X_0\subset O$ is a compact and convex set; let the $(1+n)$-dimensional function $F(t,x,y)=(f^0(t,x,y),f(t,x,y))^T$, where $f(t,x,y)=(f^1(t,x,y),...,f^n(t,x,y))^T$, be continuous on the set $[s_0,t_1]\times O^2$ and continuously differentiable with respect to $x$ and $y$; next , let $\Delta$ be a set of piecewise-continuous initial functions $\varphi(t)\in K, t\in(-\infty,s_1]$, and $q^i(t_0,\tau,x_0,x_1),\ i=\overline{0,l}$ be continuously differentiable scalar functions on the set $[s_0,s_1]\times[\tau_1,\tau_2]\times X_0\times O$.

To each element $w=(t_0,\tau,x_0,\varphi(\cdot))\in W=(s_0,s_1)\times(\tau_1,\tau_2)\times X_0\times\Delta$ we assign the delay differential equation

$$\dot{x}(t)=f(t,x(t),x(t-\tau)),t\in[t_0,t_1] \tag{1}$$

with the initial condition

$$x(t)=\varphi(t),t<t_0,\ x(t_0)=x_0. \tag{2}$$

The condition (2) is said to be the discontinuous initial condition since in general $\varphi_0(t_0-)\neq x_0$.

**Definition 1.** Let $w\in W$ . A function $x(t)=x(t;w)\in O,t\le t_1$ is called a solution corresponding to the element $w$ if it satisfies the condition (2) and is absolutely continuous on the interval $[t_0,t_1]$ and satisfies Eq. (1) almost everywhere on $[t_0,t_1]$.

**Definition 2.** An element $w\in W$ is said to be admissible if there exists the corresponding solution $x(t)=x(t;w)$ satisfying the condition

$$q^i(t_0,\tau,x_0,x(t_1))=0,i=\overline{1,l}. \tag{3}$$

We denote the set of admissible elements by $W_0$.

Now we consider the functional

$$J(w)=q^0(t_0,\tau,x_0,x(t_1))+\int_{t_0}^{t_1}f^0(t,x(t),x(t-\tau))dt,$$

where $w\in W_0, x(t)=x(t;w)$.

**Definition 3.** An element $w_0=(t_{00},\tau_0,x_{00},\varphi_0(\cdot))\in W_0$ is said to be optimal if

$$J(w_0) = \inf_{w \in W_0} J(w). \tag{4}$$

The problem (1)-(4) is called the initial data optimization problem with the discontinuous initial condition.

**Theorem 1.** Let $w_0 = (t_{00}, \tau_0, x_{00}, \varphi_0(\cdot))$ be an optimal element. Then there exist a vector $\pi = (\pi_0, ..., \pi_l) \neq 0,\ \pi_0 \leq 0$ and a solution $\Psi(t) = (\psi_0(t), \psi(t))$, where $\psi(t) = (\psi_1(t), ..., \psi_n(t)))$, of the equation

$$\begin{cases} \dot{\psi}(t) = -\Psi(t)F_x(t, x_0(t), x_0(t-\tau_0)) - \Psi(t+\tau_0)F_y(t+\tau_0, x_0(t+\tau_0), x_0(t)), \\ \Psi(t) = 0, t > t_1, \end{cases} \tag{5}$$

such that the following conditions hold:

1) the conditions for $\Psi(t) = (\psi_0(t), \psi(t))$

$$\psi_0(t) = \pi_0, t \in [t_{00}, t_1], \psi(t_1) = \pi Q_{x_1}^0,$$

where

$$Q = (q^0, ..., q^l)^T, Q_{x_1}^0 = Q_{x_1}(t_{00}, \tau_0, x_{00}, x_0(t_1));$$

2) the condition for the initial moment $t_{00}$

$$\pi Q_{t_0}^0 = \Psi(t_{00})F(t_{00}, x_0(t_{00}), \varphi_0(t_{00} - \tau_0)) + \Psi(t_{00} + \tau_0)[F(t_{00} + \tau_0, x_0(t_{00} + \tau_0), x_{00}) -$$
$$- F(t_{00} + \tau_0, x_0(t_{00} + \tau_0), \varphi_0(t_{00}))],$$

where $x_0(t) = x(t; w_0), \varphi_0(t_{00} - \tau_0) = \varphi_0(t_{00} - \tau_0 +),\ \varphi_0(t_{00}) = \varphi_0(t_{00}-);$

3) the condition for the delay parameter $\tau_0$

$$\pi Q_\tau^0 = \Psi(t_{00} + \tau_0)[F(t_{00} + \tau_0, x_0(t_{00} + \tau_0), x_{00}) - F(t_{00} + \tau_0, x_0(t_{00} + \tau_0), \varphi_0(t_{00}))] +$$
$$+ \int_{t_{00}+\tau_0}^{t_1} \Psi(t)F_y(t, x_0(t), x_0(t-\tau_0))\dot{x}_0(t-\tau_0)dt;$$

4) the condition for the initial vector $x_{00}$

$$(\pi Q_{x_0}^0 + \psi(t_{00}))x_{00} = \max_{x_0 \in X_0}(\pi Q_{x_0}^0 + \psi(t_{00}))x_0;$$

5) the condition for the initial function $\varphi_0(t)$

$$\int_{t_{00}-\tau_0}^{t_{00}} \Psi(t+\tau_0)F(t+\tau_0, x_0(t+\tau_0), \varphi_0(t))dt = \max_{\varphi(\cdot)\in\Delta} \int_{t_{00}-\tau_0}^{t_{00}} \Psi(t+\tau_0)F(t+\tau_0, x_0(t+\tau_0), \varphi(t))dt.$$

**Some Comments**. The condition 3) is essential novelty in this work. Let $\varphi_0(t) \stackrel{def}{=} \varphi_0(s_1-)$ for $t \geq s_1$ then for arbitrary $\tau \in [\tau_1, \tau_2]$ the functions $\varphi_0(t + \tau_0 - \tau)$ belongs to set $\Delta$. It is clear that, for arbitrary $\tau \in [\tau_1, \tau_2]$ we have

$$\int_{t_{00}-\tau_0}^{t_{00}} \Psi(t+\tau_0)F(t+\tau_0, x_0(t+\tau_0), \varphi_0(t))dt \geq \int_{t_{00}-\tau_0}^{t_{00}} \Psi(t+\tau_0)F(t+\tau_0, x_0(t+\tau_0), \varphi_0(t+\tau_0-\tau))dt.$$

Thus

$$\int_{t_{00}}^{t_{00}+\tau_0} \Psi(t)F(t, x_0(t), \varphi_0(t-\tau_0))dt = \max_{\tau\in[\tau_1,\tau_2]} \int_{t_{00}}^{t_{00}+\tau_0} \Psi(t)F(t, x_0(t), \varphi_0(t-\tau))dt.$$

From the condition 5) follows

$$\Psi(t+\tau_0)F(t+\tau_0, x_0(t+\tau_0), \varphi_0(t)) = \max_{\varphi\in K} \Psi(t+\tau_0)F(t+\tau_0, x_0(t+\tau_0), \varphi),$$

$t \in [t_{00} - \tau_0, t_{00}]$.

If $K$ is convex set then from the condition 5) follows

$$\int_{t_{00}-\tau_0}^{t_{00}} \Psi(t+\tau_0) F_y(t+\tau_0, x_0(t+\tau_0), \varphi_0(t))\varphi_0(t)dt =$$

$$= \max_{\varphi(\cdot)\in\Delta} \int_{t_{00}-\tau_0}^{t_{00}} \Psi(t+\tau_0) F_y(t+\tau_0, x_0(t+\tau_0), \varphi_0(t))\varphi(t)dt .$$

In this form, for the first time, the necessary condition was proved by G. Kharatishvili [1] for initial function.

The expression

$$\Psi(t_{00}+\tau_0)[F(t_{00}+\tau_0, x_0(t_{00}+\tau_0), x_{00}) - F(t_{00}+\tau_0, x_0(t_{00}+\tau_0), \varphi_0(t_{00}))]$$

is the effect of discontinuous initial condition (2).

Let $\hat{x}_0 \in O$ and $\hat{x}_1 \in O$ be fixed points. For each initial data $\sigma = (t_0, \tau, \varphi(\cdot)) \in \Sigma =$ $= (s_0, s_1) \times (\tau_1, \tau_2) \times \Delta$ we assign the differential equation (1) with the initial condition

$$x(t) = \varphi(t), t < t_0, x(t_0) = \hat{x}_0 .$$

An element $\sigma \in \Sigma$ is said to be admissible if there exists corresponding solution $x(t) = x(t;\sigma)$ satisfying the condition $x(t_1) = \hat{x}_1$.

An element $\sigma_0 = (t_{00}, \tau_0, \varphi_0(\cdot)) \in \Sigma_0$ is said to be optimal if

$$J_1(\sigma_0) = \inf_{\sigma\in\Sigma_0} J_1(\sigma),$$

where

$$J_1(\sigma) = \int_{t_0}^{t_1} f^0(t, x(t), x(t-\tau))dt, x(t) = x(t;\sigma),$$

and $\Sigma_0$ is set of admissible elements.

The theorem presented below directly follows from Theorem 1.

**Theorem 2.** Let $\sigma_0 = (t_{00}, \tau_0, \varphi_0(\cdot))$ be an optimal element. Then there exists a nontrivial solution $\Psi(t) = (\psi_0(t), \psi(t))$, $\psi_0(t) = const \le 0$, of the equation (5) such that the condition 5) holds. Moreover, the following conditions are fulfilled:

6) the condition for the initial moment $t_{00}$

$$\Psi(t_{00})F(t_{00}, x_0(t_{00}), \varphi_0(t_{00}-\tau_0)) + \Psi(t_{00}+\tau_0)[F(t_{00}+\tau_0, x_0(t_{00}+\tau_0), x_{00}) -$$
$$- F(t_{00}+\tau_0, x_0(t_{00}+\tau_0), \varphi_0(t_{00}))] = 0;$$

7) the condition for the delay parameter $\tau_0$

$$\Psi(t_{00}+\tau_0)[F(t_{00}+\tau_0, x_0(t_{00}+\tau_0), x_{00}) - F(t_{00}+\tau_0, x_0(t_{00}+\tau_0), \varphi_0(t_{00}))] +$$
$$+ \int_{t_{00}+\tau_0}^{t_1} \Psi(t) F_y(t, x_0(t), x_0(t-\tau_0))\dot{x}_0(t-\tau_0)dt = 0.$$

## 2. Equation with the continuous initial condition

To each element $\sigma = (t_0, \tau, \varphi(\cdot)) \in \Sigma$ we assign the differential equation (1) with the initial condition

$$x(t) = \varphi(t), t \le t_0, \tag{6}$$

where is assumed that $\varphi(t_0) = \varphi(t_0-)$.

The condition (6) is said to be the continuous initial condition, since always $x(t_{00})=\varphi(t_0)$.
An element $\sigma \in \Sigma$ is said to be admissible if there exists the corresponding solution $x(t)=x(t;\sigma)$ satisfying the condition

$$q^i(t_0,\tau,\varphi(t_0),x(t_1))=0, i=\overline{1,l}. \tag{7}$$

We denote the set of admissible elements by $\Sigma_1$.
An element $\sigma_0=(t_{00},\tau_0,\varphi_0(\cdot))\in\Sigma_1$ is said to be optimal if

$$J_2(\sigma_0)=\inf_{\sigma\in\Sigma_0} J_2(\sigma), \tag{8}$$

where

$$J_2(\sigma)=q^0(t_0,\tau,\varphi_0(t_0),x(t_1))+\int_{t_0}^{t_1} f^0(t,x(t),x(t-\tau))dt.$$

The problem (1), (6)-(8) is called the initial data optimization problem with the continuous initial condition.

**Theorem 3.** Let $\sigma_0=(t_{00},\tau_0,\varphi_0(\cdot))$ be an optimal element and the function $\varphi_0(t)$ be continuously differentiable in a neighborhood of the point $t_{00}$. Then there exist a vector $\pi=(\pi_0,...,\pi_l)\neq 0,\ \pi_0\leq 0$ and a solution $\Psi(t)=(\psi_0(t),\psi(t))$ of the equation (5) such that the conditions 1) and 5) hold. Moreover, the following conditions are fulfilled :

8) the condition for the initial moment $t_{00}$

$$\pi Q^0_{t_0}=\Psi(t_{00})[(0,\dot\varphi_0(t_{00}))^T-F(t_{00},x_0(t_{00}),\varphi_0(t_{00}-\tau_0))];$$

9) the condition for the delay parameter $\tau_0$

$$\pi Q^0_\tau=\int_{t_{00}+\tau_0}^{t_1}\Psi(t)F_y(t,x_0(t),x_0(t-\tau_0))\dot x_0(t-\tau_0)dt;$$

10) the condition for the point $\varphi_0(t_{00})$

$$[\pi Q^0_{x_0}+\psi(t_{00})f(t_{00},x_0(t_{00}),\varphi_0(t_{00}-\tau_0))]\varphi_0(t_{00})=$$
$$=\max_{\varphi\in K}[\pi Q^0_{x_0}+\psi(t_{00})f(t_{00},x_0(t_{00}),\varphi_0(t_{00}-\tau_0))]\varphi.$$

Theorem 1 and 2, on the basis of the variation formulas [2-4], are proved by a method given in [4].

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## References

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