BELLMAN EQUATION FOR OPTIMAL PROCESSES WITH NONLINEAR MULTI-PARAMETRIC BINARY DYNAMIC SYSTEM

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In general, nonlinear multi-parametric binary dynamic system(NMBDS) is defined as follows [1]:

$$\xi_v s(c) = F_v(c, s(c), x(c)),\ v = 1,2,...,k \tag{1}$$

$$s(c^0) = s^0 \tag{2}$$

where $c = (c_1, c_2, ..., c_k) \in G_d = \{c \mid c \in Z^k, c_1^0 \le c_1 \le c_1^{L_1}, ..., c_k^0 \le c_k \le c_k^{L_k}, c_i \in Z\}$ is a point in Z^k determining position; $L_i, i = 1,2,...,k$ where k is a positive integer, is the duration of the stage i of the process. Here, Z is the set of integers. For $s(c) \in S, x(c) \in X; S = [GF(2)]^m$, $X = [GF(2)]^r$ are state and input index (alphabet) respectively; $s(c)$ and $x(c)$ are defined over the set Z^k as an m and r dimensional state and input vectors at the point c. $c^0 = (c_1^0, c_2^0, ..., c_k^0)$ is the initial position vector of the system and s^0 is the initial state vector of the system. $c^{L_i} = (c_1^{L_i}, c_2^{L_i}, ..., c_k^{L_i})$ is the point to which the system moves after the stage $i-1$. ξ_v is a shift operator defined as follows [1]:

$$\xi_v s(c) = s(c + e_v); e_v(0,...,0,\overset{v}{1},0,...,0),\ v = 1,2,...,k. \tag{3}$$

Boolean vector functions [2,3] denoted by $F_v(\cdot) = \{F_{v_1}(\cdot), F_{v_2}(\cdot), ..., F_{v_m}(\cdot)\}$ are nonlinear functions, where $GF(2)$ is a Galois field and the representation $(\cdot)$ denotes $(c, s(c), x(c))$ for simplicity.

Optimal piecewise process represented by the system (1)-(2) is characterized by the pseudo Boolean functional [3] given by:

$$J(x) = \varphi(s(c^L) \tag{4}$$

which we use as an objective functional for the considered problem in the presented proceeding. Here, $L = L_1 + L_2 + ... + L_k$ is the time duration of this process.

Now, we can state the considered original problem represented by NMBDS as follows:
In order for a given NMBDS to go from the known initial state s^0 to any desired state $s^*(c^L)$, to which we expect to access in L steps, a control $x(c) \in \hat{X}$ [6] must exist such that the functional in (4) has a minimal value:

$$\xi_v s(c) = F_v(c, s(c), x(c)),\ c \in G_d,\ v = 1,2,...,k \tag{5}$$

$$s(c^0) = s^0 \tag{6}$$

$$x(c) \in \hat{X},\ c \in \hat{G}_d \tag{7}$$

$$J(x) = \varphi(s(c^L)) \to \min. \tag{8}$$

Since the transfer functions are Boolean and the objective functional which characterizes the process is the process is pseudo Boolean [3,5], the pseudo Boolean expressions of the transfer functions have been obtained by the operations given in [5]. After this step, the problem can be stated as follows:

$$\xi_v s(c) = \hat{F}_v(c, s(c), x(c)),\ c \in G_d,\ v = 1,2,...,k \tag{9}$$

$$s(c^0) = s^0 \tag{10}$$

$$x(c) \in \hat{X},\ c \in \hat{G}_d \tag{11}$$

$$J(x) = \varphi(s(c^L)) \to \min. \tag{12}$$

where $\hat{F}_v(\cdot)\,(v = 1,...,k)$ denotes the pseudo Boolean expression of the Boolean vector function $F_v(\cdot)\,(v = 1,...,k)$ and $\hat{G}_d = G_d \setminus \{c^L\}$.

Since we have shown that the principle of optimality [4] is satisfied for the problem (9)-(12), hereafter we can formulate this problem as an optimal problem [6]:

$$\xi_v s(c) = \hat{F}_v(c, s(c), x(c)),\ c \in G_d(\sigma),\ v = 1,2,...,k \tag{13}$$

$$s(\sigma) = \chi \tag{14}$$

$$x(c) \in \hat{X},\ c \in G_d(\sigma) \tag{15}$$

$$J(x) = \varphi(s(c^L)) \to \min \tag{16}$$

where $\chi \in S = [GF(2)]^m, \sigma \in G_d, G_d(\sigma) = \{c | \sigma_1 \le c_1 \le c_1^{L_1},..., | \sigma_k \le c_k \le c_k^{L_k}\}$. If we substitute $\sigma = c^0$ and $\chi = s^0$ into the problem (13)-(16), we can obtain the first problem stated above.

For every fixed σ and χ, let a function be corresponded to the optimal value of pseudo Boolean functional in the problem (13)-(16). We say that this function is the piecewise analogue of Bellman function [4] in the problem (9)-(12):

$$B(\sigma, \chi) = \min\ \varphi(s(c^L)). \tag{17}$$

Here, minimization is implemented on the set of admissible controls $x(c)\ (c \in G_d(\sigma))$.

We derive the Bellman equation for $B(\sigma, \chi)$ function: Assume that $x^0(c)\ (c \in G_d(\sigma))$ is the corresponding optimal control to the problem (13)-(16) with the initial condition and $s^0(c)\ (c \in G_d(\sigma))$ is also the corresponding optimal trajectory to that problem.

Let the point $\xi_v \sigma \in G_d(\sigma)\ (v = 1,2,...,k)$ and any element $y(c) \in \hat{X}$ be considered. If $x(\sigma) = y(c)$, then the state of the system in the point $\xi_v \sigma$ is determined by the following equality:

$$s(\xi_v \sigma) = F_v(\sigma, \chi, y(c)). \tag{18}$$

We consider the following problem:

$$\xi_v s(c) = F_v(c, s(c), x(c)),\ c \in G_d(\xi_v \sigma) \tag{19}$$

$$s(\xi_v \sigma) = F_v(\sigma, \chi, y(c)) \tag{20}$$

$$x(c) \in \hat{X},\ c \in G_d(\xi_v \sigma) \tag{21}$$

$$J(x) = \varphi(s(c^L)) \to \min. \tag{22}$$

If $\hat{y}(c)\ (c \in G_d(\xi_v \sigma))$ is the corresponding optimal control to the problem (19)-(22) and $\hat{s}(c)\ (c \in G_d(\xi_v \sigma))$ is also the corresponding optimal trajectory to that problem, then according to our definition stated above, the equality

$$\varphi(s(c^L)) = B(\xi_v \sigma, F_v(\sigma, \chi, y(c))) \tag{23}$$

can be obtained.

Now, let the following admissible control

$$\tilde{x}(c) = \begin{cases} y(c), & for \ \ c = \sigma \\ \hat{y}(c), & for \ c \in G_d(\xi_v \sigma) \end{cases} \tag{24}$$

be considered for the problem (13)-(16). Then $\tilde{s}(c)$ is determined by

$$\tilde{s}(c) = \begin{cases} F_v(\sigma, \chi, y(c)), & for \ \ c = \sigma \\ \hat{s}(c), & for \ c \in G_d(\xi_v \sigma) \end{cases}. \tag{25}$$

It is clear that the corresponding value of the pseudo Boolean functional $J(x) = \varphi(s(c^L))$ to the control $\hat{x}(c)$ $(c \in G_d(\sigma))$ is

$$\varphi(\tilde{s}(c^L)) = \varphi(\hat{s}(c^L)) = B(\xi_v \sigma, F_v(\sigma, \chi, y(c))) \ . \tag{26}$$

Since $\tilde{x}(c)$ $(c \in G_d(\sigma))$ is not generally optimal control, we can write

$$\varphi(\tilde{s}(c^L)) \ge \varphi(s^0(c^L)) = B(\sigma, \chi) \ . \tag{27}$$

Thus, we have

$$B(\sigma, \chi) \le B(\xi_v \sigma, F_v(\sigma, \chi, y(c))) \ . \tag{28}$$

On the other hand, if $y(c) = x^0(\sigma)$, then $\hat{y}(c)$ $(c \in G_d(\xi_v \sigma))$ equals to $x^0(c)$ $(c \in G_d(\xi_v \sigma))$ by the principle of optimality. So,

$$B(\sigma, \chi) = B(\xi_v \sigma, F_v(\sigma, \chi, x^0(\sigma))) \ . \tag{29}$$

By (28) and (29), Bellman equation can be obtained as follows:

$$B(\sigma, \chi) = \min_{y(c) \in \hat{X}} B(\xi_v \sigma, F_v(\sigma, \chi, x^0(\sigma))) \ , \ \chi \in S \ . \tag{30}$$

The initial condition for Bellman equation is given on the right-upper region of G_d and directly determined with the help of the following equality

$$B(c^L, \chi) = \varphi(\chi), \ \chi \in S \ . \tag{31}$$

Hence, Bellman function is the solution of the equation (30) with the initial condition (31).
It is clear that Bellman equation (30) has exact solution.

If η_v is inverse operator of the shift operator ξ_v, then we have

$$\eta_v B(\xi_v \sigma, \chi) = B(\sigma, \chi) . \tag{32}$$

Thus, Bellman equation (30) can be derived as follows:

$$\eta_v B(\xi_v \sigma, \chi) = \min_{y(c) \in \hat{X}} B(\xi_v \sigma, \hat{F}_v(\sigma, \chi, y(c))) \ , v = 1,2,...,k \ . \tag{33}$$

Substituting $\xi_v \sigma = \delta$ in (33), we obtain

$$\eta_v B(\delta, \chi) = \min_{y(c) \in \hat{X}(\eta_v \delta)} B(\delta, \hat{F}_v(\eta_v \delta, \chi, y(c))) \ , v = 1,2,...,k \ . \tag{34}$$

If $\eta_v(\eta_{v'} B(\delta, \chi))$ is evaluated for every $v, v' = 1,2,...,k$, then

$$\eta_v(\eta_{v'} B(\delta, \chi)) = \eta_v \big(\min_{y(c) \in \hat{X}(\eta_{v'} \delta)} B(\delta, \hat{F}_{v'}(\eta_{v'} \delta, \chi, y(c))) \big)$$

$$= \min_{y(c) \in \hat{X}(\eta_{v'} \delta)} \eta_v B(\delta, \hat{F}_{v'}(\eta_{v'} \delta, \chi, y(c)))$$

$$= \min_{y(c)\in \hat{X}(\eta_v \eta_{v'}\delta)} [\min_{x(c)\in \hat{X}(\eta_v \delta)} B(\delta, \hat{F}_v(\eta_v \delta, F_{v'}(\eta_{v'}\eta_v \delta, y(c), \chi), x(c)))] . \quad (35)$$

Similarly,

$$\eta_{v'}(\eta_v B(\delta,\chi)) = \min_{y(c)\in \hat{X}(\eta_{v'}\eta_v \delta)} [\min_{x(c)\in \hat{X}(\eta_{v'} \delta)} B(\delta, \hat{F}_{v'}(\eta_{v'} \delta, F_{v}(\eta_{v}\eta_{v'} \delta, y(c), \chi), x(c)))] . \quad (36)$$

The condition implying the existence of the unique solution of the system of equations (13) is given by [1]

$$F_v(c+e_\mu, F_\mu(c,s(c),x(c)), x(c+e_\mu)) = F_\mu(c+e_v, F_v(c,s(c),x(c)), x(c+e_v)), \quad v,\mu = 1,2,...,k. \quad (37)$$

Substituting $\mu = v'$ in (37), we obtain

$$\hat{F}_v(\eta_v\delta, \hat{F}_{v'}(\eta_v\eta_{v'}\delta, \chi, x(\eta_v\eta_{v'}\delta)), x(\eta_v\delta)) = \\ = \hat{F}_{v'}(\eta_{v'}\delta, \hat{F}_{v}(\eta_{v'}\eta_{v}\delta, \chi, x(\eta_{v'}\eta_{v}\delta)), x(\eta_{v'}\delta)),\ v,v' = 1,2,...,k. \quad (38)$$

By (35), (36) and (38), the following result is derived:

$$\eta_v(\eta_{v'}B(\delta,\chi)) = \eta_{v'}(\eta_v B(\delta,\chi)),\ v,v' = 1,2,...,k .$$

This result is the condition implying the existence of the exact solution for Bellman equation (30). If Bellman equation (30) is solved subject to the condition (31) over the curve $\hat{L}(c^0, c^1, ..., c^L)$, then we achieve the following functions after L steps:

$$B(c^L,\chi),\ B(c^{L-1},\chi), ..., B(c^0,\chi) .$$

$B(c^0, s(c^0))$ is the minimal value of the pseudo Boolean functional in the problem (13)-(16). The optimal control is determined with the help of the following condition:

$$B(\xi_v c, \hat{F}_v(c, s(c), x^0(c))) = \min_{x(c)\in\hat{X}} B(\xi_v c, \hat{F}_v(c, s(c), x(c))) .$$

where $c \in \hat{L}(c^0, c^1, ..., c^L)$. Here, $\hat{L}(c^0, c^1, ..., c^L)$ is piecewise curve associating the point c^0 with the point c^L [1]. v takes value such that $\xi_v c \in \hat{L}(c^0, c^1, ..., c^L)$. Then, the optimal trajectory is determined by

$$\xi_v s(c) = \hat{F}_v(c, s(c), x^0(c)),\ v = 1, 2, ..., k$$
$$s(c^0) = s^0 .$$

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