ON ONE GAUSS TYPE QUADRATURE SCHEME FOR NUMERICAL SOLUTION OF LIPPMAN-SCHWINGER EQUATIONS

Jemal Sanikidze[1] and Manana Mirianashvili[2]

Niko Muskhelishvili Institute of Computational Mathematics, Tbilisi, Georgia
[1]*j_sanikidze@yahoo.com*, [2]mmirianashvili@yahoo.com

The subject of consideration of the present paper is construction of a certain approximate scheme for numerical solution of Lippman-Scwinger singular integral equations known in nuclear physics (see, e.g.[1-2])

$$\varphi(x;x_0)+\int_0^{+\infty}\frac{N(x,y)\varphi(y;x_0)dy}{y^2-x_0^2}=N(x,x_0)\quad \left(x_0\in(0,+\infty)\right), \tag{1}$$

Solution of the indicated equation is supposed for this or that given values of the parameter of singularity x_0, at this the corresponding singular integral is considered in the Cauchy principal value sense. Structure of the kernels (potentials) is determined by the data of the initial concrete problem and in practical cases their calculation often requires application of numerical experiment. This circumstance evidently increases importance of application of possibly more accurate quadrature formulas to approximation of the corresponding integrals in equation (1). We note here also that in the theory of such equations, certain concrete potentials, whose structure and properties at arbitrary variation of the variables x, y in the interval $(0,+\infty)$ are characteristic in a known sense for a number of other classes of potentials, play a significant role. From this point of view we may note as an example the known in literature potential of Yukawa

$$N(x,y)=\frac{v}{4}\ln\frac{(y+x)^2+\eta^2}{(y-x)^2+\eta^2}\quad\left(v=-\frac{50}{41.47},\eta=0.7\right),$$

also the so-called Reid's potential representing a certain linear combination of the similar logarithmic expressions with given concrete coefficients. We note that the structure (differential properties) of such potentials having even such relatively simple form, complicates construction of more or less effective in the sense of accuracy numerical algorithms, which besides certain difficulties in the corresponding theoretical considerations, is confirmed by calculation practice also.

From the existing calculation schemes in this direction we firstly apparently may mention the scheme from [3], with the help of which some numerical results (in the case of the mentioned above potential of Reid) are given ibid. Further, there are also works [4,5] dealing with investigation of accuracy and questions of solvability of calculation schemes for such equations. Note that consideration of the integral in (1) in the form

$$\int_0^{+\infty}\frac{N(x,y)\varphi(y;x_0)-N(x,x_0)\varphi(x_0;x_0)}{y^2-x_0^2}dy$$

and further, its reduction to an integral with finite limits with the help of one or another substitution is a base for the above mentioned schemes (evidently, influence of calculation errors in the values of $N(x,y)$ when applying quadrature formulas to the schemes based on such transformations may turn out to be more perceptible in the error of the result).

In this paper a concrete numerical scheme of different structure for approximate solution of the equation (1), based on direct application of the Gauss quadrature formulas is offered (after reducing to an integral with finite limits). Possibility of application of such formulas to singular integrals with Cauchy type kernels is well known nowadays and can be realized under the condition that the points of singularity belong to a concrete discrete set of points (see e.g. [6]). In the case considered here, having integrals with knowingly fixed singularities, the Gauss accuracy, as it is shown further, is realized introducing a numerical parameter which is determined in the way indicated below. Note that one of such schemes using Chebyshev's knots is offered in the paper [7]. Although the mentioned scheme holds a certain simplicity, clarification of the question of improvement of the error estimate $O(n^{-1}\ln n)$ of the corresponding quadrature scheme given there, encounters difficulties. Below, a considerably convenient from our viewpoint calculation scheme for numerical solution of the equation (1) with likewise Gauss accuracy rate is stated with respect to which more accuracy rate $O(n^2 \ln n)$ ($n \to \infty$) is developed. At this it is important also that the corresponding scheme has significantly convenient structure among similar schemes from the viewpoint of its practical realization. Proceeding to construction of the mentioned scheme we consider in the concerned integral $y = y(t) = c\left(-1 + 1/t^2\right)$ $(0 < t \le 1)$, where the constant c $(c > 0)$ is supposed to be arbitrary for a while. Using afterwards a number of transformations, the corresponding integral is brought to the form

$$\frac{2}{c+x_0}\int_0^1 \frac{t\,dt}{\left(t^2 - \dfrac{c}{c+x_0}\right)\left[t^2\left(1-\dfrac{x_0}{c}\right)-1\right]}.$$

Further, we transform the obtained integral applying the identity

$$\frac{t}{t-\sqrt{\dfrac{c}{c+x_0}}} = \frac{1}{\sqrt{\dfrac{c}{c+x_0}}}\left(\frac{t^2}{t-\sqrt{\dfrac{c}{c+x_0}}} - t\right).$$

In the issue the initial integral is reduced to an expression involving the following difference of two integrals (one of which is singular and the other is regular):

$$\int_0^1 \frac{t^2 N(x,y(t))\varphi(y(t);x_0)dt}{\left(t^2 - \dfrac{c}{c+x_0}\right)\left[t^2\left(1-\dfrac{x_0}{c}\right)-1\right]} - \int_0^1 \frac{tN(x,y(t))\varphi(y(t);x_0)dt}{\left(t + \sqrt{\dfrac{c}{c+x_0}}\right)\left[t^2\left(1-\dfrac{x_0}{c}\right)-1\right]}. \tag{2}$$

For the sake of brevity the first integral in (2) can be written in the form

$$\int_0^1 \frac{N(x,y(t))t^2\varphi(y(t);x_0)F(t,x_0,c)dt}{t-\sqrt{\dfrac{c}{c+x_0}}}, \tag{3}$$

where

$$F(t,x_0,c) = \left\{\left(t + \sqrt{\frac{c}{c+x_0}}\right)\left[t^2\left(1-\frac{x_0}{c}\right)-1\right]\right\}^{-1}.$$

Integral (3) has singularity with kernel $t-\sqrt{\dfrac{c}{c+x_0}}$ for arbitrary $c > 0$. At this introduction of the parameter c enables us to pass from the initial integral with fixrd singularity in equation (1), to a singular integral with movable singularity, which gives us possibility to apply quadrature formulas to singular integrals (in the sense remarked about the results in [6]). For any fixed x_0 this is realized by proper choice of the free parameter c. Noting this firstly we will proceed to construction of the proposed scheme. Herewith we will pay a principal attention to singular integral (3). Considering below n even, we apply a quadrature formula

$$\int_0^1 \frac{N(x,y(t))t^2\varphi(y(t);x_0)F(t,x_0,c)dt}{t-\sqrt{\frac{c}{c+x_0}}} \approx \sum_{k=1}^{n} \frac{A_{kn}N(x,y(t_{kn}))t_{kn}^2\varphi(y(t_{kn});x_0)F(t_{kn},x_0,c)dt}{t_{kn}-\sqrt{\frac{c}{c+x_0}}}, \quad (4)$$

to this integral (for a while formally). Here $\{t_{kn}\}_{k=1}^n$, $\{A_{kn}\}_{k=1}^n$ are the knots (the zeros of the Legendre polynomial) and coefficients of the Gauss quadrature formula respectively in the interval (0,1). According to the results mentioned in [6], formula (4) has the highest ($2n$ in the given case) algebraic accuracy rate, when the corresponding point of singularity $\sqrt{c/(c+x_0)}$ belongs to the set of zeros of the second kind Legendre function [([8], $6.9)], corresponding to the given n (in the present case we should note that these zeros belong to interval (0,1)).

Taking into account mentioned above, we get convinced that the structure of the construction of approximate scheme stated here gives possibility to use the simplest of them _ (rational) zero $1/2$. Due to it, the corresponding parameter c is determined from the condition $\sqrt{c/(c+x_0)} = 1/2$, from which we get $c = x_0/3$. Thereby we come to possibly more simple Gauss quadrature formula in the singular case.

In the second (regular) integral firstly we will use identity $\frac{t^2}{t+\varepsilon_n} + r_n(t;\varepsilon_n) = t$, $r_n(t;\varepsilon_n) = t\left(1-\frac{t}{t+\varepsilon_n}\right)$, where it is supposed that $\varepsilon_n = O(1/n^2)$ $(n\to\infty)$. Taking this into account, after having changed the factor t by $t^2/(t+\varepsilon_n)$ in the numerator of the corresponding integrand, we will use the Gauss quadrature formula as well to the mentioned integral (with the same knots $\{t_{kn}\}_{k=1}^n$ and coefficients $\{A_{kn}\}_{k=1}^n$).

The rate of the practically realizable accuracy, as it is usually at numerical solution of integral equations by a quadrature method, is specified by differential properties (order of differentiability) of densities of approximable integrals in (2). The basic investigations from this viewpoint are connected with the expression $N(x,y(t))t^2\varphi(y(t);x_0)$. Considering details concerning to this question, we will at this restrict ourselves by the case of the mentioned above Yukawa's potential.

Transforming the integral in (1) according to the scheme stated here, with the help of equalization of the corresponding approximating expression for $x = c(-1+1/\tau^2)$ $(0<\tau\le 1)$ to $N(x(\tau),x_0)$ we come to a certain functional equation (which finally can be reduced to a linear algebraic system with respect to values of the unknown function φ). Noticing that in the quadrature sum the values of the function φ at knots t_{kn} are involved at the values t_{kn}^2 and multiplying the corresponding functional equation by τ^2, the obtained equation can be considered as an equation with respect to function $t^2\varphi$. The sense of such approach, first of all, is connected with development of the interesting for us convergence rate of the calculation scheme. At this we note that in the corresponding problems the final goal is determination of the value (for the given values of x_0)

$$-arctg\frac{\varphi(x_0;x_0)}{x_0}\quad (0<x_0<+\infty),$$

called a phase of nucleon-nucleon interaction. In the given case having the approximate values $\{t^2\varphi\}(t_{kn})$ found from the corresponding system, we can determine approximately the values $\tau_0^2\varphi(x_0;x_0)$ $\left(\tau_0^2 = 1/{x_0+1}\right)$ and consequently, the corresponding values of the unknown phase in the form

$$-arctg\frac{\tau_0^2\varphi(x_0;x_0)}{1-\tau_0^2}\quad (0<\tau_0<1).$$

We will now stop at the question of convergence of the scheme (which, according to the above said, in the given case consists in study of behavior of derivatives of the function $N(x(\tau),y(t))t^2\varphi(y(t);x_0)$ with respect to t).

Generally the basic question is behavior of the considered derivatives in the neighborhood of $t\to 0$. For this purpose we can use the asymptotic estimates $\varphi(x;x_0)=O(1/x)$, $\varphi\,'(x;x_0)=O(1/x^2)$ $(x\to+\infty)$ obtained in [4]. Besides, on the basis of considerations developed in [4] concerning the indicated estimates, the estimate $\varphi\,''(x;x_0)=O(1/x^2)$ $(x\to+\infty)$ can be obtained as well.

Using these estimates and remembering the remark about ε_n, we can assert validity of estimate $O\left(\ln n/{n^2}\right)$ ($n\to\infty$) with respect to the convergence rate of the offered scheme.

The research was supported by the GNSF grant (grant No. /ST08/3-390).

References

1. J.E. Brawn, E.D. Jackson. Nucleon-nucleon interaction. (in Russian) "Atomizdat", Moscow (1979) 248 p.
2. J Tailor. Theory of dispersion. (in Russian) "Mir", Moscow (1975) 566 p.
3. M.I.Hartel, F. Tabakin. Nuclear Saturation and Smoothness of Nucleon-Nuclton Potentials. Nuclear Physics, A158, Holland, Amsterdam (1970) 1-42 p.
4. J. Sanikidze. On the Problem of Quadrature Approximation of One Singular Integral Operator. Comput. Methods in Appl. Math. (2001) v.1, N 2, pp.199-210.
5. D.G. Sanikidze. On the Numerical Solution of a Class of Singular Integral Equations on an infinite Interval. Differential Equations. V. 41, № 9 (2005) pp. 1353-1358.
6. V.V. Panasiuk, M.P. Savruk, A.P. Datsishin. Distribution of Stresses Along Cracks in Plates and Shells. (in Russian) "Naukova Dumka", Kiev (1976) 444 p.
7. D.G. Sanikidze. On Gauss Quadrature Formulas for Cauchy Type Integrals with Fixed Singularities and Some of Their Applications. (in Russian) Proc. of IV International Scientific-Technical Conference "Analytic and Numerical Methods of Modelling of Natural and Social Problems", Penza (2009) pp.60-63.
8. G. Sege. Ortogonal polynomials. (in Russian) "Fizmatgiz", Moscow (1962) 500 p.