INVESTIGATION OF SYSTEM MAP|M|1|RQ BY THE METHOD OF ASYMPTOTICAL SEMIINVARIANTS TO THE THIRD ORDER

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The modern world develops fast rates. It is already impossible for itself to present a reality without an information technology. They extend practically everywhere also participate almost in all spheres of human activity.

One of the most important characteristics of a network of data transmission is the size of a delay necessary for delivery of the message from a source to destination which in networks of casual access is irregular.

In real-world systems, there often occurs to effect of retrial attempts to get service, and collisions require consideration of models that are beyond the framework of classical queueing systems. Therefore, there is growing interest to the analysis of such real-word systems. An alternative approach is applying the asymptotic analysis method to such systems, which allows one to find the asymptotic probability distribution if the number of requests in a retrial pool.

By the asymptotic analysis method in queueing theory, we call solving equations that describe certain characteristics of a system under certain limiting condition; the form of such a condition should be specified for particular models and investigation problems.

In the present paper we consider a retrial system with a conflict of notifications and input MAP-flow; to study it, we propose a method of asymptotic semiinvariants of an third order.

As a mathematical model of this system, we consider single-server queueing system with retrial pool (RQ) fed by a MAP-flow of requests, controlled by Markov chain, given the matrix Q of infinitesimal characteristics $q_{k\nu}$, a set of nonnegative variables $\lambda_k \ge 0$ and set of probabilities $d_{k\nu}$ for $k \ne \nu$. If a request finds the server is free, it occupies it for a service time, which is exponentially distributed with parameter μ. If the server is busy, then the arriving request and request under service collide and are sent to a retrial pool. The requests stay in the retrial pool for a delay time, which is exponentially distributed with parameter σ. After a random delay, each request from the retrial pool makes a retrial attempt to capture the server.

Let $i(t)$ be the number of requests in the retrial pool, and let $l(t)$ define the state of the server as follows:

$$l(t)=\begin{cases}0, \mathit{if}\ \ \mathit{the\ server\ is\ free}, \\ 1, \mathit{if}\ \ \mathit{the\ server\ is\ busy}.\end{cases}$$

Denote by

$$P\{\, l(t)=l, k(t)=k, i(t)=i\}=P(l,k,i,t)$$

the probability that the server at time t is in state l, controlled Markov chain is in state k and there are i requests in retrial pool.

The state change process $\{l(t), k(t), i(t)\}$ of the described system is a Markov process.

For probability distribution $P(l,k,i,t)$ of state $\{l,k,i\}$ of described RQ-system find system of Kolmogorov's differential equations for the stationary regime:

$$\begin{cases}-(\lambda_k+i\sigma)P(0,k,i)+\mu P(1,k,i)+\lambda_k P(1,k,i-2)+(i-1)\sigma P(1,k,i-1)+ \\ +\sum\limits_{\nu}\{P(0,\nu,i)(1-d_{\nu k})+P(1,\nu,i-2)d_{\nu k}\}q_{\nu k}=0, \\ -(\lambda_k+\mu+i\sigma)P(1,k,i)+\lambda_k P(0,k,i)+\sigma(i+1)P(0,k,i+1)+ \\ +\sum\limits_{\nu}\{P(1,\nu,i)(1-d_{\nu k})+P(0,\nu,i)d_{\nu k}\}q_{\nu k}=0.\end{cases} \tag{1}$$

From the above system, we obtain a system of equations for the function

$$H(l,k,u) = \sum_i e^{jui} P(l,k,i), \tag{2}$$

which can be rewritten in a matrix form:

$$j\sigma \frac{\partial H(u)}{\partial u} A(ju) = H(u)B(ju), \tag{3}$$

where denoting the row vector $H(u) = \{H(0,u), H(1,u)\}$, and the block matrices $A(ju),\ B(ju)$ are

$$A(ju) = \begin{pmatrix} -I & e^{-ju}I \\ e^{ju}I & -I \end{pmatrix}, \qquad B(ju) = \begin{pmatrix} Q-Y & Y \\ \mu I + e^{2ju}Y & Q-Y-\mu I \end{pmatrix}. \tag{4}$$

The matrices admit the following expansions:

$$A(ju) = \sum_{\nu=0}^{\infty} \frac{(ju)^{\nu}}{\nu!} A_{\nu}, \quad B(ju) = \sum_{\nu=0}^{\infty} \frac{(ju)^{\nu}}{\nu!} B_{\nu}.$$

The obtained equality (3) calls the basic equation for the mathematical model in question.

First-Order Asymptotic

To find the first-order asymptotic, we make the following changes of variables in the basic equations (3):

$$\sigma = \varepsilon, \quad u = \varepsilon w, \quad H(u) = F_1(w,\varepsilon),$$

then equation (3) takes the form

$$j \frac{\partial F_1(w,\varepsilon)}{\partial w} A(j\varepsilon w) = F_1(w,\varepsilon) B(j\varepsilon w). \tag{5}$$

Passing the limit as $\varepsilon \to 0$ in equation (5), we find

$$j \frac{\partial F_1(w)}{\partial w} A_0 = F_1(w) B_0, \tag{6}$$

where $F_1(w) = \lim_{\varepsilon \to 0} F_1(w,\varepsilon)$.

We seek for a solution $F_1(w)$ to the system (6) in the form of a product

$$F_1(w) = R\Phi(w). \tag{7}$$

The vector R as the meaning of the probability distribution of values of the process $l(t)$ as $\varepsilon \to 0$. To find scalar function $\Phi_1(w)$, we add together all equations of system (5) and multiply this equality by the all-one vector E. Then, expanding the matrices $A(j\varepsilon w)$, $B(j\varepsilon w)$ in the small parameter and substituting into the obtained equality the product (7), we get nonlinear equation

$$j \frac{\partial \Phi(w)}{\partial w} RA_1E = \Phi_1(w) RB_1E, \tag{8}$$

whose solution $\Phi_1(w)$ satisfying the initial condition $\Phi_1(0) = 1$, is of the form

$$\Phi_1(w) = \exp\{jw\kappa_1\}, \tag{9}$$

where κ_1 can be found as follows.

By substituting (9) into (7), and then into (6), we obtain an nonlinear scalar system of equations

$$R(B_0 + \kappa_1 A_0) = 0, \tag{10}$$

which defines a vector $R = R(\kappa_1)$, satisfying the normalizing condition $RE = 1$.

By substituting (9) into equation (8), we obtain an nonlinear scalar equation with respect to κ_1

$$R(\kappa_1)(B_1 + \kappa_1 A_1)E = 0. \tag{11}$$

Second-Order Asymptotic

To find the second-order asymptotic in the equation (3) we make the changes

$$H(u) = H_2(u)\exp\left\{\frac{ju}{\sigma}\kappa_1\right\},$$

for unknown vector function $H_2(u)$, we obtain an equation

$$j\sigma\frac{\partial H_2(u)}{\partial u}A(ju) = H_2(u)\{B(ju) + \kappa_1 A(ju)\}, \tag{12}$$

in which we make the change of variables

$$\sigma = \varepsilon^2, \quad u = \varepsilon w, \quad H_2(u) = F_2(w,\varepsilon). \tag{13}$$

For equation (12) we find

$$j\varepsilon\frac{\partial F_2(w,\varepsilon)}{\partial w}A(j\varepsilon w) = F_2(w,\varepsilon)\{B(j\varepsilon w) + \kappa_1 A(j\varepsilon w)\}. \tag{14}$$

Passing the limit as $\varepsilon \to 0$ in this equation, where $\lim\limits_{\varepsilon\to 0} F_2(w,\varepsilon) = F_2(w)$, we find

$$F_2(w)\left(B_0 + \kappa_1 A_0\right) = 0. \tag{15}$$

We seek for a solution $F_2(w)$ to this system in the form

$$F_2(w) = R\Phi_2(w) = R\,\exp\left\{\frac{(jw)^2}{2}\kappa_2\right\}, \tag{16}$$

where R is vector is solution of the system (10) and normalizing condition $RE = 1$, and scalar function $\Phi_2(w)$ is obtained from (16), where κ_2 can be found as follows.

Taking the power series expansions of the matrices $A(j\varepsilon w),\ B(j\varepsilon w)$ in the parameter ε, we rewrite the system (14) with accuracy to infinitesimal terms of order ε^2 as follows:

$$j\varepsilon\frac{\partial F_2(w,\varepsilon)}{\partial w}A_0 = F_2(w,\varepsilon)\{B_0 + \kappa_1 A_0 + j\varepsilon w\left(B_1 + \kappa_1 A_1\right)\} + O(\varepsilon^2). \tag{17}$$

We represent the solution $F_2(w,\varepsilon)$ to this system in the form

$$F_2(w,\varepsilon) = \exp\left\{\frac{(jw)^2}{2}\kappa_2\right\}\{R + j\varepsilon w f_1(w)\} + O(\varepsilon^2). \tag{18}$$

Substituting the product (18) into previous equation, we obtain a non-uniform system of equations

$$f_1(w)(B_0 + \kappa_1 A_0) + R\left(B_1 + \kappa_1 A_1 + \kappa_2 A_0\right) = 0$$

with respect to vector function $f_1(w)$. From this system we can rewrite vector function $f_1(w)$ as expansion

$$f_1(w) = G_1(w)R + h_1, \tag{19}$$

where $G_1(w)$ is any scalar function, and vector h_1 is the decision of system

$$h_1(B_0 + \kappa_1 A_0) + R\left(B_1 + \kappa_1 A_1 + \kappa_2 A_0\right) = 0.$$

Let's present h_1 in the form of decomposition

$$h_1 = g_1 + \kappa_2 g,$$

where

$$g_1(B_0 + \kappa_1 A_0) + R\left(B_1 + \kappa_1 A_1\right) = 0,$$

$$g(B_0 + \kappa_1 A_0) + RA_0 = 0. \tag{20}$$

To find the scalar parameter κ_2, we sum up all equations of system (14), right multiplying it by the all-one vector E, we obtain

$$j\varepsilon\frac{\partial F_2(w,\varepsilon)}{\partial w}A(j\varepsilon w)E = F_2(w,\varepsilon)\{B(j\varepsilon w) + \kappa_1 A(j\varepsilon w)\}E. \tag{21}$$

Expanding the matrices $A(j\varepsilon w)$, $B(j\varepsilon w)$ into (21) in the parameter ε and considering $R(B_1+\kappa_1 A_1)E=0$ and (20), we obtain the solution κ_2 in the form

$$\kappa_2=-\frac{g_1\left(B_1+\kappa_1 A_1\right)E+\frac{1}{2}R\left(B_2+\kappa_1 A_2\right)E}{RA_1E+g\left(B_1+\kappa_1 A_1\right)E}. \qquad (22)$$

Third-Order Asymptotic

To find the third-order asymptotic in the equation (12) we make the changes

$$H_2(u)=H_3(u)\exp\left\{\frac{(ju)^2}{2\sigma}\kappa_2\right\},\sigma=\varepsilon^3,\quad u=\varepsilon w,\quad H_3(u)=F_3(w,\varepsilon). \qquad (23)$$

Rewrite equation (12) in the form

$$j\varepsilon^2\frac{\partial F_3(w,\varepsilon)}{\partial w}A(j\varepsilon w)=F_3(w,\varepsilon)\left\{B(j\varepsilon w)+\kappa_1 A(j\varepsilon w)+\kappa_2 j\varepsilon wA(j\varepsilon w)\right\}. \qquad (24)$$

Further, operating similar as in the second-order asymptotic, we receive the equality defining parameter κ_3:

$$\kappa_3=-\frac{h_1\{B_2+\kappa_1 A_2+2\kappa_2 A_1\}E+g_2\left(B_1+\kappa_1 A_1\right)E+\frac{1}{3}R\left(B_3+\kappa_1 A_3+3\kappa_2 A_2\right)E}{RA_1E+g\left(B_1+\kappa_1 A_1\right)E},$$

where

$$h_1\{B_0+\kappa_1 A_0\}+R\{\kappa_2 A_0+B_1+\kappa_1 A_1\}=0,$$
$$g_2(B_0+\kappa_1 A_0)+R\left(B_2+\kappa_1 A_2+2\kappa_2 A_1\right)+2h_1\left(B_1+\kappa_1 A_1+\kappa_2 A_0\right)=0,$$
$$g(B_0+\kappa_1 A_0)+RA_0=0.$$

Thus, in work the method asymptotic semiinvariants for research of mathematical model of RQ-system is offered. The mathematical model of the given system for a stationary case is offered and considered. The method asymptotic semiinvariants in a condition of a growing delay time in retrial pool to the third order is applied.

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