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LINEAR HARA FRONTIERS IN PORTFOLIO OPTIMIZATION

Süleyman Özekici[1], Ethem Çanakoğlu[2]

[1]Koç University, İstanbul Turkey
[2]Warwick Business School, Coventry, UK
[1]*sozekici@ku.edu.tr*, [2]*ethem.canakoglu@wbs.ac.uk*

1 Introduction

One of the most important problems faced by investors involve the allocation of their wealth among different investment opportunities in a market consisting of risky assets. Determination of optimal portfolios is a rather complex problem depending on the objective of the investor. In this setting, the objective of the investor is to maximize the expected value of a utility function of the terminal wealth. The risk preferences of the investor is given and measured by the utility function. The most widely used measures of risk-aversion were introduced by Pratt [7] and Arrow [1].

In most of the multiperiod problems, the rates of return of the assets during consecutive periods are assumed to be uncorrelated. In a realistic setting, this is not correct and the dependence among the rates of return in consecutive periods should also be considered. This dependence or correlation is often achieved through a stochastic market process that affects all deterministic and probabilistic parameters of the model. A tractable and realistic approach is provided by using a Markov chain that represents the economic, financial, social, political and other factors which affect the returns of the assets. The use of a modulating stochastic process as a source of variation in the model parameters and of dependence among the model components has proved to be quite useful in operations research and management science applications. In finance literature the Markov modulated market is described as regime switching. Hamilton [6] provides one of the earlier papers that suggest the use of regime switching to explain business cycles. He suggests that the state of the business can be described by a state variable which can be parameterized as a first-order Markov process. Gray [5] uses the regime switching approach for short term interest rates. He first examines different models with both single and multiple regimes, and proposes a new model called generalized regime switching. He concludes that the generalized regime switching model outperforms simple single-regime models in an out-of-sample forecasting experiment. Çakmak and Özekici [2] applied the idea to multiperiod mean-variance portfolio optimization problem. Considering a market with one riskless and some risky assets, a multiperiod mean-variance formulation is developed.

In this paper we will summarize the results for exponential utility function from HARA class. Similar results exist for power and logarithmic utility functions and details for these results with numerical analysis can be found in Çanakoğlu and Özekici [4].

2 The Stochastic Market Model

Suppose that the state of the market in period n is denoted by Y_n and $Y = \{Y_n; n = 0,1,2,\cdots\}$ is a Markov chain with a discrete state space E and transition matrix Q. Let $R(i)$ denote the random vector of asset returns in any period given that the stochastic market is in state i. The means, variances and covariances of asset returns depend only on the current state of the stochastic market. The market consists of one riskless asset with known return r_f and standard deviation $\sigma_f(i) = 0$, and m risky assets with random returns $R^n(i) = (R_1^n(i), R_2^n(i), ..., R_m^n(i))$ in period n if the state of the market is i. We assume that the random returns in consecutive periods are conditionally independent given the market states.

Moreover, $R^n(i)$ and $R^k(i)$ are independent and identically distributed random vectors whenever $k \neq n.$ This implies that the distributions of the asset returns depend only on the state of the market independent of time. For this reason, we will let $R(i) = R^n(i)$ denote the random return vector in any period n to simplify our notation.

We let $r_k(i) = E[R_k(i)]$ denote the mean return of the k th asset in state i and $\sigma_{kj}(i) =$ $\mathrm{Cov}(R_k(i), R_j(i))$ denote the covariance between k th and j th asset returns in state i. The excess return of the k th asset in state i is $R_k^e(i) = R_k(i) - r_f$. Our notation is such that r_f is a scalar and $r(i)$, $r^e(i)$ are column vectors for all i. We let X_n denote the amount of investor's wealth at period n and the vector $u = (u_1, u_2, \cdots, u_m)$ denotes the amounts invested in risky assets $(1,2,..., m)$.

A utility function U is a non-decreasing real valued function defined on the real numbers. Hyperbolic absolute risk aversion (HARA) is described by the absolute risk aversion function $U''(x)/U'(x) = 1/(a + bx)$ where HARA utility functions with an identical parameter b belong to the same class. This also implies that the risk tolerance function of the investor, defined as the inverse of the risk aversion function, has the linear form $a + bx$ for HARA utility functions. We assume that the utility of the investor depends on the market state so that the utility function is $U(i, x)$ if the state of the market is i and the wealth is x at the terminal time. The HARA class consists of exponential, logarithmic, and power utility functions.

3 Exponential Utility Function

We assume that the utility of the investor in state i is given by the exponential function

$$U(i, x) = K(i) - C(i)\exp(-x/\beta) \tag{1}$$

with $\beta > 0,\ C(i) > 0$ where we can easily see that Pratt-Arrow's measure of absolute risk aversion is simply equal to the constant $-U''(i, x)/U'(i, x) = 1/\beta$ for all i. The exponential utility function is one of the most widely used ones to represent investors attitude towards risk in portfolio optimization. It has constant absolute risk aversion given by $1/\beta$ which means that the investor has the same risk preferences for random outcomes independent of his wealth.

Theorem 1 *Let the utility function of the investor be the exponential function (1) and suppose that the riskless asset return does not depend on the market state. Then, the optimal solution for the value function is*

$$v_n(i, x) = K_n(i) - C_n(i)e^{-x/\beta_n} \tag{2}$$

and the optimal portfolio is

$$u_n^*(i, x) = \alpha(i)\beta_{n+1} \tag{3}$$

where

$$\beta_n = \frac{\beta}{r_f^{T-n}},\ K_n(i) = Q^{T-n}K(i),\ C_n(i) = \hat{Q}^{T-n}C(i) \tag{4}$$

and

$$\hat{Q}(i, j) = Q(i, j)E\left[\exp(-R^e(i)'\alpha(i))\right] \tag{5}$$

for all $n = 0,1,\cdots, T-1;$ *and* $\alpha(i)$ *satisfies*

$$E\left[R_k^e(i)\exp(-R^e(i)'\alpha(i))\right] = 0 \tag{6}$$

for all assets $k = 1,2,\cdots, m$ *and all* i.

Using the well-known dynamic programming algorithm Theorem 1 can be proven. Detailed proof and the analysis can be found in Çanakoğlu and Özekici [3]. In Theorem 1, we have found a closed-form solution for the optimal portfolio. We can further characterize the optimal policy by noting from (6) that the optimal solution satisfies

$$E\left[\left(R_k(i)-r_f\right)\exp\left(-\left(R(i)-r_f\right)'\alpha(i)\right)\right]=0$$

which implies

$$\frac{E\left[R_k(i)\exp\left(-R(i)'\alpha(i)\right)\right]}{E\left[\exp\left(-R(i)'\alpha(i)\right)\right]}=r_f \tag{7}$$

for all assets $k=1,2,\cdots,m.$ A significant characterization implied by the optimal solution (3) is that the optimal distribution of wealth invested on the risky assets depend only on the state of the market independent of time. Moreover, it is quite amazing that it is also independent of the wealth level. If the market is in state i in period n, then the total amount of money invested on the risky assets does not depend on the current wealth level x. Moreover, the proportion on wealth allocated for asset k is

$$w_k(i)=\frac{\alpha_k(i)}{\sum_{k=1}^{m}\alpha_k(i)} \tag{8}$$

which is totally independent of both time n and wealth x. The exponential investor therefore decides by considering the state of the market only. The intuition in this amazing result is in the exponential utility function. Like the memorylessness property of the exponential distribution that is associated with time, the exponential utility function implies a similar property associated with the wealth of the investor. The investor is memoryless in the sense that his current wealth level does not affect how he chooses to allocate his money among the risky assets. However, note that there is randomness involved in this choice due to the randomly changing market conditions. Our results are of course consistent with similar work in the literature on exponential utility functions, but our stochastic market approach makes our model more realistic without causing substantial difficulty in the analysis. Another important observation is that the structure of the optimal portfolio is not affected by the transition matrix Q of the stochastic market. It only depends on the joints distribution of the risky asset returns as prescribed by (7). This further implies that the exponential investor is not only memoryless about his wealth, he is also myopic since he does not care much about future states of the market in choosing his portfolios.

The evolution of the wealth process X using the optimal policy can be analyzed by the wealth dynamics equation

$$X_{n+1}=r_f X_n+r_f^{n+1-T}A(Y_n)\beta$$

where we define random variables $A(i)=R^e(i)'\alpha(i)=\sum_{k=1}^{m}\alpha_k(i)R_k^e(i)$ for any state i with mean $\overline{\alpha}(i)=\sum_{k=1}^{m}\alpha_k(i)r_k^e(i)$. Using the optimal allocation of funds to the assets the optimal wealth process can be written as

$$X_n=r_f^n X_0+r_f^{n-T}\beta\sum_{k=0}^{n-1}A(Y_k). \tag{9}$$

It can be shown that both the mean and the standard deviation of X_T depend linearly on β. This shows that the exponential frontier is the straight line

$$E_i\left[X_T\right]=r_f^T x_0+\left(\frac{m(i,T)}{v(i,T)}\right)\mathrm{SD}_i\left(X_T\right) \tag{10}$$

where $\mathrm{SD}_i\left(X_T\right))=\sqrt{\mathrm{Var}_i\left(X_T\right)}$, $m(i,T)=\sum_{k=0}^{T-1}Q^k\overline{\alpha}(i)$, and

$$v^2(i,T) = \sum_{k=0}^{T-1}\sum_{m=0}^{T-1} Cov_i\left(A(Y_k), A(Y_m)\right)$$

In other words, the expected value and standard deviation of the terminal wealth fall on this straight line when they are calculated and plotted for different values of β. Also, it cuts the zero-risk level at $E_i[X_T] = r_f^T x_0$ as expected. The reason for this is that for zero-risk level the investor puts all of his money on the riskless asset. The return of the riskless asset until the terminal time T is r_f^T, and the wealth at the terminal time will be $r_f^T x_0$ for sure. The risk premium, or Sharpe ratio, for the exponential investor is given by $m(i,T)/v(i,T)$.

The distribution of the final wealth other than just the mean and variance is also important. To derive this distribution we can find the Fourier transform

$$E_i[\exp(j\gamma X_T)] = \exp\left(j\gamma r_f^T x_0\right) E_i\left[\exp\left(j\gamma\beta\sum_{k=0}^{T-1} A(Y_k)\right)\right]$$

of the final wealth using the Fourier transform $\Phi_i(z) = E\left[\exp\left(jz'R^e(i)\right)\right]$ and the fact that

$$E\left[\exp\left(j\gamma\beta\sum_{k=0}^{T-1} A(Y_k)\right)\middle| Y_0, Y_1, \ldots, Y_{T-1}\right] = \prod_{k=0}^{T-1} E[\exp(j\gamma\beta A(Y_k)) \mid Y_k] = \prod_{k=0}^{T-1} \Phi_{Y_k}(\gamma\beta\alpha(Y_k)). \tag{11}$$

Therefore, the Fourier transform of the final wealth can be written as

$$E_i[\exp(j\gamma X_T)] = \exp\left(j\gamma r_f^T x_0\right)\Phi_i(\gamma\beta\alpha(i)) \tag{12}$$

$$\cdot \sum_{i_1, i_2, \ldots, i_{T-1}} Q(i, i_1) Q(i_1, i_2) \cdots Q(i_{T-2}, i_{T-1}) \prod_{k=1}^{T-1} \Phi_{i_k}(\gamma\beta\alpha(i_k)).$$

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