ON THE PRICING OF A EUROPEAN OPTION

Petre Babilua[1] and Besarion Dochviri[2]

[1]I. Javakhishvili Tbilisi State University, Tbilisi, Georgia, *petre_babilua@yahoo.com*
[2]I. Javakhishvili Tbilisi State University, Tbilisi, Georgia, *b.dochviri@math.sci.tsu.ge*

1. We consider the financial (B,S)-market consisting only of two assets: a bank account (bonds) $B=(B_n)$ and stocks (shares) $S=(S_n)$, $n=0,1,\dots,N$. According to the well-known Cox-Ross-Rubinstein discrete model, the time-dependent behavior (evolution) of the variables B_n and S_n is defined by the recurrent equalities

$$B_n=(1+r)B_{n-1},\quad B_0>0, \tag{1}$$

$$S_n=(1+\rho_n)S_{n-1},\quad S_0>0. \tag{2}$$

It is assumed that the family $\{\mathsf{P}\}$ of probability measures *P* is defined on the measurable space $(\Omega,\mathsf{F},\mathsf{F}_n)$, $n=0,1,\dots,N$ by filtration [1].

In equalities (1), (2), $r>0$ is an interest rate, and ρ_n for any probability measure $P\in\mathsf{P}$ is a sequence of independent, identically distributed random variables taking only two values *a* and *b*; also, $P(\rho_n=b)=p$, $P(\rho_n=a)=1-p$, $-1<a<r<b$ [1]-[3].

Let us now assume that there is some investor who has the initial capital $X_0=x>0$ and wants to get the capital f_N in the future by using the capability of the (B,S)-market. In that case, we deal with the so-called investment problem.

2. Let the price of one bond B_0 and the price of one stock is S_0 at the initial moment of time. Suppose that at the moment of time $n=0$ the investor purchased β_0 quantity of bonds and γ_0 quantity of stocks. Therefore the investor's initial capital can be written in the form

$$X_0=X_0^\pi=\beta_0B_0+\gamma_0S_0, \tag{3}$$

where $\pi=\pi_0=(\beta_0,\gamma_0)$ is said to form the investor's portfolio or strategy at the moment of time $n=0$.

Let us now assume that there is a sequence of F_{n-1}-measurable functions $g=(g_n)$, $n=0,1,\dots,N$, $g_0=0$. Suppose that before the arrival of the moment of time $n=1$, the investor transformed his portfolio $\pi_0=(\beta_0,\gamma_0)$ to the new portfolio $\pi_1=(\beta_1,\gamma_1)$ in a manner such that the equality

$$X_0^\pi=\beta_1B_0+\gamma_1S_0+g_1 \tag{4}$$

is satisfied. Thus if $g_1\ge 0$, then the initial capital X_0^π diminishes by the value g_1; if $g_1\le 0$, then X_0^π increases by the value g_1.

After the arrival of the moment of time $n=1$, the investor will have the capital

$$X_1^\pi=\beta_1B_1+\gamma_1S_1, \tag{5}$$

where B_1 and S_1 are the new prices of one bond and one stock, respectively, at the moment of time $n=1$.

Analogously, for any moments of time $n-1$ and *n* we have

$$X_{n-1}^\pi=\beta_{n-1}B_{n-1}+\gamma_{n-1}S_{n-1}, \tag{6}$$

$$X_{n-1}^{\pi} = \beta_n B_{n-1} + \gamma_n S_{n-1} + g_n , \qquad (7)$$

$$X_n^{\pi} = \beta_n B_n + \gamma_n S_n . \qquad 8)$$

The strategy $\pi = (\pi_n) = (\beta_n, \gamma_n)$ is called a (x, f, N)-hedge if

$$X_0^{\pi} = X_0 = x,$$

$$X_N^{\pi} \ge f_N ,$$

where $f = f_N = f_N(S_0, S_1, \ldots, S_N)$ is some payoff function.

If we have the equality $X_N^{\pi} = f_N$, then π is called a minimal hedge.

For $X_0 = x > 0$ and $f = f_N$ we denote by $\Pi(x, f, N)$ the set of all (x, f, N)-hedges.

Now let us define a standard European call option. This is a derivative (secondary) security with the payoff function

$$f = f_N = (S_N - K)^+ = \max(S_N - K, 0). \qquad (9)$$

The owner of this option enjoys the right to buy a stock at a price *K* at a certain moment of time *N*. If $S_N > K$, then the owner of the option will buy a stock at a price *K*, sell it at once at a price S_N and have a gain

$$f_N = S_N - K .$$

His gain will actually be equal to

$$f_N = S_N - K - C_N ,$$

where C_N is the so-called fair (rational) price of a standard European call option. If $S_N \ge K$, then the owner of the option will not carry out the operation with his option and his loss will be equal to C_N.

The problem of the investor (option seller) consists in the following: using the fair price of the option

$$C_N = \inf\{x > 0 : \Pi(x, f, N) \ne \varnothing\}$$

it is required to construct a minimal hedge $\pi_n^* = (\beta_n^*, \gamma_n^*)$. In other words, the investor's capital must be equal to f_N at a moment of time *N*.

The basic problems of the pricing of a European option can be formulated as follows:

1) defining a fair price C_N;

2) constructing a minimal hedge $\pi_n^* = (\beta_n^*, \gamma_n^*)$;

3) constructing the investor's capital process $X_n^{\pi^*}$ for the strategy π_n^*.

3. Let us consider the financial (B, S)-market (1), (2) and nonself-financed strategies π_n. Assume that the sequence of $\mathcal{F}_{n-1}$-measurable functions $g = (g_n)$ defined by the equality

$$g_n = c_1 \beta_n B_{n-1} + c_2 \gamma_n S_{n-1} \qquad (10)$$

is given, where the constants c_1 and c_2 are such that $0 < c_1 < 1$, $0 < c_2 < 1$.

Theorem 3. *Assume that the financial market* (1), (2) *is considered and the sequence of* $\mathcal{F}_{n-1}$*-measurable functions* $g = (g_n)$ *is given by means of* (10). *Then*

1) *the fair price* C_N *of a European type option with the execution at the moment of time N and the payoff function* $f_N = f_N(S_0, S_1, \ldots, S_N)$ *is defined by the formula*

$$C_N = E^*\left[\left(\frac{1-c_1}{1+r}\right)^N \cdot f_N\right],$$

where E^* *is the averaging with respect to a measure* $P^* \in \mathcal{P}$ *such that*

$$P^*(\rho_n = b) = p^*, \quad P^*(\rho_n = a) = 1 - p^*, \quad 0 < p^* < 1,$$

$$p^* = \frac{r + c_1(1+a) - c_2(1+r) - a}{(b-a)(1-c_1)};$$

2) *there exists a minimal* (x, f, N)*-hedge* $\pi^* = (\pi_n^*) = (\beta_n^*, \gamma_n^*)$, $n = 0,1,\dots,N$, *whose* $\mathcal{F}_{n-1}$*-measurable components are defined by the formulas*

$$\beta_n^* = \frac{X_{n-1}^* - \gamma_n^* S_{n-1}(1-c_2)}{B_{n-1}(1-c_1)},$$

$$\gamma_n^* = \frac{\alpha_n^* B_n}{S_{n-1}(1-c_1)},$$

where $\alpha_k^* = \alpha_k^*(\rho_1,\dots,\rho_{k-1})$, $k \geq 2$, $\alpha_1^* = \text{const}$, *are the definite* $\mathcal{F}_{n-1}$*-measurable functions;*

3) *the capital* $X^{\pi^*} = \left(X_n^{\pi^*}\right)$, $n = 0,1,\dots,N$, *corresponding to the hedge* $\pi^* = (\pi_n^*)$ *is given by the formula*

$$X_n^{\pi^*} = E^*\left[\left(\frac{1-c_1}{1+r}\right)^{N-n} \cdot f_N \middle| \mathcal{F}_n\right].$$

References

1. B. N. Shiryaev, Fundamental principles of stochastic financial mathematics.(Russian) *Fakti, Modeli, Teoriya*, v. I-II, *Fazis, Moscow*, 1998.
2. T. Abuladze, B. Dochviri, On the pricing of European options. *Bull. Georgian Acad. Sci.* 169 (2004), No. 1, 13-15.
3. T. Abuladze, P. Babilua, B. Dochviri, M. Shashiashvili, On the modeling of the European option pricing theory. *Reports of Enlarged Sessions of the Seminar of I. Vekua Institute of Applied Mathematics* 21 (2006), No. 3, 5-8.