ON THE CENTRAL LIMIT THEOREM FOR WEAKLY DEPENDENT RANDOM VARIABLES WITH VALUES IN D[0,1]

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Let $\{X_n(t), t\in[0,1], n\geq 1\}$ be a sequence of random variables with values in $D[0,1]$ (a space of all real-valued, right continuous and with left limits functions on $[0,1]$ which is endowed with the Skorohod topology). We say that $\{X_n(t), t\in[0,1], n\geq 1\}$ satisfies a central limit theorem if the distribution of $\frac{1}{\sqrt{n}}\left(X_1(t)+...+X_n(t)\right)$ weakly converges to a Gaussian distribution in $D[0,1]$. The central limit theorem in $D[0,1]$ is very important from application point of view. It immediately implies asymptotic normality of empirical and weighted empirical processes. The central limit theorem for the sequence of independent identically distributed (i.i.d.) random variables with values in $D[0,1]$ were studied by many authors (see [1-5] and references therein). The first central limit theorem was proved by Hahn [4]. Her result can be formulated as following.

Theorem A. Let $\{X_n(t), t\in[0,1], n\geq 1\}$ be a sequence of i.i.d. random variables with values in $D[0,1]$ such that

$$EX_1(t)=0, \quad EX_1^2(t)<\infty \text{ for all } t\in[0,1] \tag{1}$$

Assume that there exist nondecreasing continuous functions G and F on $[0,1]$ and numbers $\alpha>\frac{1}{2}$, $\beta>1$ such that for all $0\leq s\leq t\leq u\leq 1$ the following two conditions hold:

$$E\left(X_1(u)-X_1(t)\right)^2\leq\left(G(u)-G(t)\right)^{\alpha},$$

$$E\left(X_1(u)-X_1(t)\right)^2\left(X_1(t)-X_1(s)\right)^2\leq\left(F(u)-F(s)\right)^{\beta}. \tag{2}$$

Then $\{X_n(t), t\in[0,1], n\geq 1\}$ satisfies the CLT in $D[0,1]$ and the limiting Gaussian process is sample continuous.

As it was already noticed in [2] the condition (2) is connected with the fourth moments of the process $X_1(t)$.This condition does not allow us to apply Theorem A to a wide class of weighted empirical processes.

Later Theorem A was improved in series papers (see [1-3, 5]). In [2] authors obtained the following results.

Theorem B. Let $\{X_n(t), t\in[0,1], n\geq 1\}$ be a sequence of i.i.d. random variables with values in $D[0,1]$ satisfying the condition (1) and assume that there exist nondecreasing continuous functions G and F on $[0,1]$ and numbers $\alpha,\beta>0$ such that for all $0\leq s\leq t\leq u\leq 1$ the following two conditions hold:

$$E\left(X_1(u)-X_1(t)\right)^2\leq\left(G(u)-G(t)\right)^{1/2}\log^{-4,5-\alpha}\left(1+\left(G(u)-G(t)\right)^{-1}\right), \tag{3}$$

$$E\left(\left|X_1(t)-X_1(s)\right|\wedge 1\right)^2\left(X_1(u)-X_1(t)\right)^2\leq\left(F(u)-F(s)\right)\log^{-5-\beta}\left(1+\left(F(u)-F(s)\right)^{-1}\right). \tag{4}$$

Then $\{X_n(t), t\in[0,1], n\geq 1\}$ satisfies the CLT in $D[0,1]$ and the limiting Gaussian process is sample continuous.

Theorem C. The statement of Theorem B remains true if conditions (3) and (4) are replaced by

$$E\left(X_1(u)-X_1(t)\right)^2 \leq \left(G(u)-G(t)\right)^{1/2}\log^{-2,5-\alpha}\left(1+\left(G(u)-G(t)\right)^{-1}\right),$$

$$E\left(X_1(t)-X_1(s)\right)^2\left(X_1(u)-X_1(t)\right)^2 \leq \left(F(u)-F(s)\right)\log^{-5-\beta}\left(1+\left(F(u)-F(s)\right)^{-1}\right).$$

This investigation was motivated by Theorem B and our main goal is to prove the central limit theorem for the sequence $\{X_n(t), t\in[0,1], n\geq 1\}$ of weakly dependent random variables with values in $D[0,1]$. As a measure of weak dependence we use the mixing coefficients. For a given sequence $\{X_n(t), t\in[0,1], n\geq 1\}$ of $D[0,1]$ – valued random variables mixing coefficients are defined as following:

$$\rho(n)=\sup\left\{\frac{\left|E(\xi-E\xi)(\eta-E\eta)\right|}{E^{\frac{1}{2}}(\xi-E\xi)^2 E^{\frac{1}{2}}(\eta-E\eta)^2} : \xi\in L_2\left(F_1^k\right), \eta\in L_2\left(F_{n+k}^{\infty}\right), k\in N\right\}.$$

where $F_a^b-\sigma$-field generated by $X_a(t),......,X_b(t)$ and $L_2\left(F_a^b\right)-$a space of all square integrable and F_a^b – measurable random variables.

We say that $\{X_n(t), t\in[0,1], n\geq 1\}$ is $\rho-$mixing if $\rho(n)\to 0$ as $n\to\infty$.

Denote $S_n(t)=X_1(t)+...+X_n(t)$.

The next theorem is our main result.

Theorem D. Let $\{X_n(t), t\in[0,1], n\geq 1\}$ be a strictly stationary sequence of $\rho-$mixing random variables with values in $D[0,1]$ such that

$$EX_1(t)=0,\ \ E\left|X_1(t)\right|^{3+\varepsilon}<\infty \text{ for all } t\in[0,1] \text{ and some } \varepsilon>0.$$

Assume that there exist nondecreasing continuous functions $G_i(t)$ $(i=1,2,3)$ on $[0,1]$ such that for all $0\leq s\leq t\leq u\leq 1$ and some $\varepsilon_1>0$ the following hold:

$$E\left(X_1(u)-X_1(t)\right)^2\leq\left(G_1(u)-G_1(t)\right)\log^{-(4+\varepsilon+\varepsilon_1)}\left(1+\left(G_1(u)-G_1(t)\right)^{-1}\right),$$

$$E\left(\left|X_1(t)-X_1(s)\right|\wedge 1\right)^{1+\varepsilon}\left(X_1(u)-X_1(t)\right)^2\leq\left(G_2(u)-G_2(s)\right)\log^{-(4+\varepsilon+\varepsilon_1)}\left(1+\left(G_2(u)-G_2(s)\right)^{-1}\right),$$

$$\max\left(E\left|X_1(u)-X_1(t)\right|^3, E\left|X_1(u)-X_1(t)\right|^{3+\varepsilon}\right)\leq\left(G_3(u)-G_3(t)\right)\log^{-(4+\varepsilon+\varepsilon_1)}\left(1+\left(G_3(u)-G_3(t)\right)^{-1}\right),$$

$$\lim_{n\to\infty} ES_n^2(t)=\infty \text{ for all } t\in[0,1],$$

$$\sum_{k=1}^{\infty}\rho\left(2^k\right)<\infty.$$

Then $\{X_n(t), t\in[0,1], n\geq 1\}$ satisfies the central limit theorem and the limiting mean-zero, sample continuous Gaussian process has covariance function:

$$F\left(t_1,t_2\right)=\lim_{n\to\infty}\frac{1}{n}ES_n(t_1)S_n(t_2),\quad t_1,t_2\in[0,1].$$

We note that the central limit theorem for the weighted triangular array of $D[0,1]$-valued mixing random variables proved in [6] does not imply our result.

References

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