SOME LIMIT THEOREMS FOR I.I.D. AND CONDITIONALLY INDEPENDENT RANDOM VARIABLES

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Being encouraged by an excellent textbook [6], we give here limit theorems, more or less known or completely new ones, which are or may be applicable in actuarial and financial modeling.

1. Let us consider a sequence $X_0, X_1, \dots$ of i.i.d. *m*-dimensional random vectors on a probability space (Ω, F, P) such that $EX_0 = 0,\ \mathrm{cov}(X_0) = R,\ \mathrm{sp}R = \sigma^2 < \infty$. For $0 < A < 1$ and m = 1 the so called *discounted sum* $\eta_A = \sum_{j=0}^{\infty} A^j X_j$ may be interpreted as the *present value* of the consecutive payments $X_0, X_1, \dots$ with the *discount factor A*. In [2] Gerber proved that the normed sum $(1-A^2)^{1/2}\eta_A$ tends weakly to $N(0,\sigma^2)$ when $A \to 1^-$.

In the case m >1 and an $m \times m$-matrix valued A the random vector η_A, where $A^0 = I, A^{j+1} = AA^j, j \ge 1,$ and I stands for the identity $m \times m$-matrix, may have a lot of similar or other interpretations. Recently in [8] the following result was proved.

Theorem 1. If for fixed R and $c, 1 \le c < \infty$, A takes its values in the set of $m \times m$-matrices $\{A : \|A\| < 1, A = A^T, AR = RA, \|I - A\| \le c(1 - \|A\|)\}$ and $A \to I$ then $(1-A^2)^{1/2}\eta_A$ tends weakly to $N(0,R)$.

The theorem covers the case of positive scalar matrices $A=aI$ (with 0< a <1 and c=1) and diagonal ones with at least two different diagonal elements $a_{\max}$ and $a_{\min}$, in which case $c \ge (1-a_{\min})/(1-a_{\max}) > 1$, both (the latter is not less than $1 - c(1-a_{\max})$) tending to 1 from the left.

When several discount matrices are chosen periodically, we have the following assertion formulated in the case of the above-mentioned scalar ones.

Corrolary. If $B_1 = b_1 I, \dots, B_k = b_k I$ and each $b_l \to 1^-$, then for the discounted sum $\eta_B = \sum_{j=0}^{\infty} A_j^j X_j$, where $A_j = B_l$ as $j \equiv (l-1) \bmod k, l = 1, \dots, k,$ the normed sum $\left[\sum_{l=1}^{k} b_l^{2(l-1)} (1 - b_l^{2k})^{-1}\right]^{-1/2} \eta_B$ converges weakly to $N(0,R)$.

2. Second problem we deal with is connected with a product of random variables, which e,g. is applicable when investigating reinvestment problem [5].

Consider a stationary two-component sequence (ξ_j, X_j), $j = 1,2,\ldots$, where ξ_j takes its values in $\{1,\ldots,s\}$ and X_j is a real random variable; denote

$$\xi = (\xi_1, \xi_2, \ldots), \quad \xi_{1n} = (\xi_1, \ldots, \xi_n), \quad X = (X_1, X_2, \ldots), \quad X_{1n} = (X_1, \ldots, X_n).$$

One says that *X* is a *sequence of conditionally independent random variables controlled by a sequence* ξ if for any natural *n* and given ξ_{1n} the conditional distribution of X_{1n} is the direct product of conditional distributions of X_j given only the corresponding ξ_j, $j = 1,\ldots,n$, i.e., it equals $\Pi_{\xi_1} \times \cdots \times \Pi_{\xi_n}$, where Π_i is the conditional distribution of X_1 given $\{\xi_1 = i\}$, $i = 1,\ldots,s$ (see, e.g., [1,7]). For *s*=1 X becomes a sequence of i.i.d. random variables with Π_1 as a common distribution

When the random variables X_j are positive consider the *product* $T_n = X_1 \cdot \ldots \cdot X_n$ in the case when the controlling sequence ξ is a regular Markov chain for which $\{1,\ldots,s\}$ is the only ergodic class. A limiting behavior of this product is easy to describe using limit theorems for sums of such summands, called usually as random variables defined on the Markov chain (we mention here the works by Ibragimov and Linnik (1965), Aleshkevichus (1966), O'Brien (1974), Koroliuk and Turbin (1976), Grigorescu and Oprisan (1976), Sirazhdinov and Formanov (1978), Silvestrov (1982), Anisimov (1982), Bokuchava (1984) and others; for the exact references see , e.g. [1]).

Let $\pi_i = P\{\xi_1 = i\}$, $i = 1,\ldots,s$, be a common distribution of ξ_j s, $Z = (z_{il}), i,l = 1,\ldots,s$, be the fundamental matrix of the Markov chain ξ. Denote

$$\mu_i = E(\ln X_1 \mid \xi_1 = i), \sigma_i^2 = E[(\ln X_1 - \mu_i)^2 \mid \xi_1 = i], i = 1,\ldots,s,$$

$$\mu = E \ln X_1 = \sum_{i=1}^{s} \pi_i \mu_i, \sigma_0^2 = \sum_{i=1}^{s} \pi_i \sigma_i^2, t = \sum_{i,l=1}^{s} (\pi_i z_{il} + \pi_l z_{li} - \pi_i \delta_{il} - \pi_i \pi_l) \mu_i \mu_l$$

and let $N(x \mid 0, b)$ be the (0, *b*)- normal distribution function.

Theorem 2. If $\sigma_0^2 < \infty$ then for *x*>0 the following assertions concerning weak convergence hold as $n \to \infty$:

$$1)\ P\left\{ [e^{-\sum_{j=1}^{n} \mu_{\xi_j}} T_n]^{1/\sqrt{n}} < x \mid \xi_{1n} \right\} \to N(\ln x \mid 0, \sigma_0^2)\ P\text{-a.s}$$

$$2)\ P\left\{ [e^{-\sum_{j=1}^{n} \mu_{\xi_j}} T_n]^{1/\sqrt{n}} < x \right\} \to N(\ln x \mid 0, \sigma_0^2)\ ;$$

$$3)\ P\left\{ (e^{-n\mu} T_n)^{1/\sqrt{n}} < x \right\} \to N(\ln x \mid 0, \sigma_0^2 + t).$$

Example. Let X_j, *j*=1,2,…, be i.i.d. positive random variables and ν_p be an independent on this sequence geometric random variable with a parameter *p*. In [5] motivated by the interpretation of T_{ν_p} as the *total return after continued reinvestment* in the same type of business with equal breakoff probability at each step, the distributions of this and related products are studied, particularly for $p \to 0$.

Instead of independent environment let us consider the environment described by the above mentioned stationary Markov chain with *s*=2 states and the transition matrix

$\begin{pmatrix} 1-c & c \\ d & 1-d \end{pmatrix}$, where $0 < d \le 1$, $0 < c \le 1, c+d < 2$; when $c+d=2$, the chain reduces to alternating sequence and for $c+d=1$ to the independent Bernoulli sequence. For this chain $\pi_1 = d/(c+d), \pi_2 = c/(c+d)$. Let the corresponding conditonal distributions Π_1, Π_2 be the uniform ones in $[0, \alpha]$ and $[0, \beta]$, resp.,$0 < \alpha < \beta \le 1$.Thus $\mu_1 = \ln\alpha - 1, \mu_2 = \ln\beta - 1, \sigma_1^2 = \sigma_2^2 = 1 = \sigma_0^2, \mu = \pi_1 \ln\alpha + \pi_2 \ln\beta - 1$ and as it follows from [4, Ch. IV] $t = cd(2-c-d)(c+d)^{-3}\ln^2(\alpha/\beta)$. For T_n Theorem 2 holds with these concrete $\mu, \sigma_0^2 = 1, t$. Note that for alternating sequence $t = 0$ and $\mu = \ln\sqrt{\alpha\beta} - 1$.

When $s=1$ and the common distribution of i.i.d. X_js is the uniform one in [0,1], the random variable $-\ln T_n$ has the Erlang distribution with parameters n and 1 [3], which is approximated by Theorem 2 $(t=0)$.

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