PROBLEMS OF SINGULAR PERTURBATIONS IN ANALYSIS OF STOCHASTIC SYSTEMS

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Asymptotical analysis of stochastic models of systems which are considered in a random medium, that is, the evolution of stochastic system is developed under the influence of random factors. The feature of an interaction between a system and a random medium is a unilateral affect of a random medium. The local characteristics of a system change with the change of states of a random medium. This particular feature of interaction is unified by the effective mathematical methods of analysis based on the problems of singular perturbation for reducible-invertible operators.

Stochastic models of systems are determined by two processes: a switched process describing the evolution of a system, and a switching process describing the changes of a random medium.

It is assumed that the evolution of a system possesses a semi-group property and the random medium has an ergodic property.

Enumerated properties of stochastic systems extract a class of systems represented by random evolution, as an operator-valued stochastic process in a Banach space.

Effective mathematical tools of analysis are based on the problems of singular perturbation for reducible-invertible operators and on martingale characterization of Markov processes (see [1-4] and references there).

Stochastic systems are considered in the series scheme with some small series parameter $\varepsilon > 0$ and also, with two scales of time: real time for a system, and rapid time for a switching process.

The diverse scheme of asymptotical analysis of stochastic systems can be reduced to the problem of singular perturbation of a reducible – invertible operator, which can be formulated in the following way. For a given vector $\psi \in B$ the asymptotic solution

$$\varphi^{\varepsilon} = \varphi + \varepsilon\varphi_1$$

of the equation

$$\left[\varepsilon^{-1}Q + Q_1\right]\varphi^{\varepsilon} = \psi + \upsilon^{\varepsilon}$$

is constructed with the asymptotically negligible term θ^{ε} :

$$\left\|\theta^{\varepsilon}\right\| \to 0 \text{ as } \varepsilon \to 0 .$$

The generator Q of uniformly ergodic Markov process possesses the reducibly invertible property.

The Banach space B can be represented as the direct sum

$$B = N_Q \oplus R_Q \tag{1}$$

of the null-space $N_Q := \{\varphi : Q\varphi = 0\}$ and of the space of values $R_Q := \{\varphi : Q\psi = \varphi\}$. Decomposition (1) means that there exists the projector Π onto null-space N_Q and the potential operator R_0 defined by the following relation

$$R_0 := \left[Q + \Pi\right]^{-1} - \Pi ,$$

satisfying the following properties:

$$QR_0 = R_0Q = \mathrm{I} - \Pi , \; \Pi R_0 = R_0\Pi = \emptyset .$$

That is, the potential R_0 is a reducible inverse operator to the operator Q. The general solution of equation

$$Q_{\varphi} = \psi$$

can be represented as follows:

$$\varphi = R_0\psi + \varphi_0 , \quad \varphi_0 \in N_Q .$$

There exist many situations which cannot be classified. Meanwhile, it is possible to extract some logically complete variants.

The classification of problems of singular perturbation is based on properties of a contracted operator $\widehat{Q}_1$ determined by the following relation

$$\widehat{Q}_1\Pi = \Pi Q_1 \Pi .$$

The contracted operator $\widehat{Q}_1$ acts on the contracted null-space $\widehat{N}_Q$.

There are three logically complete variants:

(i) $\widehat{Q}_1$ be nonzero: $\widehat{Q}_1 \neq \emptyset$;

(ii) $\widehat{Q}_1$ is zero-operator: $\widehat{Q}_1\widehat{\varphi} = 0$ for all $\widehat{\varphi} \in \widehat{N}_Q$;

(iii) $\widehat{Q}_1$ is reducible – invertible: there exists null-space $\widehat{N}_{\widehat{Q}_1} \subset \widehat{N}_{\widehat{Q}}$ such that

$$\widehat{N}_Q = \widehat{N}_{\widehat{Q}_1} \oplus \widehat{R}_{\widehat{Q}_1} .$$

There exists also the potential operator $\widehat{R}_0 := \left[\widehat{Q}_1 + \widehat{\Pi}\right]^{-1} - \widehat{\Pi}$, where $\widehat{\Pi}$ is the projector onto $\widehat{N}_{\widehat{Q}_1}$ which is defined by the following relation

$$\widehat{\Pi}\widehat{\varphi} = \widehat{\varphi}\widehat{1}, \ \widehat{\varphi} \in \widehat{N}_{\widehat{Q}_1} .$$

The solutions of singular perturbation problems in these three variances are given in the following three propositions.

Proposition 1. Let the contracted operator $\widehat{Q}_1$ be nonzero. Then the asymptotic representation

$$\left[\varepsilon^{-1}Q + Q_1\right](\varphi + \varepsilon\varphi_1) = \psi + \theta^{\varepsilon}$$

can be realized by the relations

$$\widehat{Q}_1\widehat{\varphi} = \widehat{\psi} , \theta^{\varepsilon} = \varepsilon\theta_1 .$$

Proposition 2. Let the contracted operator $\widehat{Q}_1$ be a zero-operator: $\widehat{Q}_1\widehat{\varphi} = 0$, $\forall \widehat{\varphi} \in \widehat{N}_Q$ Let in addition the operator $Q_0 = Q_2 - Q_1 R_0 Q_1$ after contraction on the space $\widehat{N}_{\widehat{Q}}$ be nonzero. Then the asymptotic representation

$$\left[\varepsilon^{-2}Q + \varepsilon^{-1}Q_1 + Q_2\right]\left(\varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2\right) = \psi + \theta^{\varepsilon}$$

can be realized by the following relations

$$\widehat{Q}_0\widehat{\varphi} = \widehat{\psi} , \ \theta^{\varepsilon} = \varepsilon\theta_2 .$$

Proposition 3. Let the contracted operator $\widehat{Q}_1$ be reducible – invertible with null-space $\widehat{N}_{\widehat{Q}_1} \subset \widehat{N}_Q$, defined by the projector $\widehat{\Pi}$. Let the twice contracted operator $\widehat{Q}_2$ on $\widehat{N}_{\widehat{Q}_1}$ defined by the relation

$$\widehat{\widehat{Q}}_2\widehat{\Pi} = \widehat{\Pi}\widehat{Q}_2\widehat{\Pi} , \ \widehat{Q}_2\Pi = \Pi Q_2 \Pi ,$$

be nonzero.

Then the asymptotic representation

$$\left[\varepsilon^{-2}Q + \varepsilon^{-1}Q_1 + Q_2\right]\left(\varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2\right) = \psi + \theta^{\varepsilon}$$

can be realized by the following relations:

$$\widehat{\widehat{Q}}_2\widehat{\varphi} = \widehat{\psi} , \ \theta^{\varepsilon} = \varepsilon\theta_3$$

Moreover, there exists more complicated situation of singular perturbation established of the combination of considered above facts.

Analysis of stochastic systems is considered for the phase merging scheme for a Markov process on a splitting phase space, for the dynamical system with rapid Markov switchings

$$du^{\varepsilon}(t)/dt = C(u^{\varepsilon}(t), x(t/\varepsilon));$$

in the phase average scheme; for the dynamical system with accelerated Markov switchings

$$du^{\varepsilon}(t)/dt = C^{\varepsilon}(u^{\varepsilon}(t), x(t/\varepsilon^2))$$

in the diffusion approximation scheme and for the dynamical system with sharply accelerated Markov switchings

$$du^{\varepsilon}(t)/dt = C^{\varepsilon}(u^{\varepsilon}(t), x^{\varepsilon}(t/\varepsilon^3))$$

in the diffusion approximation scheme with merging and averaging.

References

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