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ANALYTICAL-NUMERICAL SOLUTIONS FOR THE ONE DIMENSIONAL PBL TURBULENCE MODEL

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1. Introduction

Analytical and propagator numerical methods are elaborated for solution of Weng-Taylor turbulence model ([1]), that was originally outlined by J. Smagorinsky (see [2-4]). In the Weng-Taylor model the eddy viscosity coefficient nonlinearly depend on velocities and is defined from additional phenomenological consideration, which constitutes a turbulence closure. In such type models sharp vertical boundary layers causes difficulties for traditional numerical methods. In this work a new numerical method is proposed, which is based on analytical representation of Weng-Taylor model solutions. It is shown that these analytical solutions of constituted initial boundary value problem can be resolved by additional solutions of system of ordinary differential equations. This system of equations is solved analytically, by using polynomial type substitutions for generalized Lagrangian variables. The obtained numerical solution, precision and effectiveness are compared to solution by using numerical propagator method ([5]).

2. Problem formulation

Weng-Taylor model equations ([1]) for horizontal U and V velocity components, written here as the functions of the vertical coordinate z , are

$$\frac{\partial U}{\partial t}=\frac{\partial}{\partial z}\left(K_m\frac{\partial U}{\partial z}\right)+Tf\cos(\alpha)\left(V-V_g\right),\ 0<z<1,\ 0<t\le T, \tag{1}$$

$$\frac{\partial V}{\partial t}=\frac{\partial}{\partial z}\left(K_m\frac{\partial V}{\partial z}\right)+Tf\cos(\alpha)\left(U_g-U\right),\ 0<z<1,\ 0<t\le T, \tag{2}$$

where $f=10^{-4}$ (Hz) is the Coriolis force frequency and $U_g=10$ (m/s), $V_g=0$ m/s. With the initial and boundary conditions:

$$U(0,z)=u_0(z),\ \ V(0,z)=\vartheta_0(z),\ \ 0\le z\le 1, \tag{3}$$

$$U(t,0)=0,\ \ V(t,0)=0,\ \ 0\le t\le T, \tag{4}$$

$$U(t,1)=V_g,\ \ V(t,1)=0,\ \ 0\le t\le T. \tag{5}$$

The eddy viscosity coefficient K_m is defined from additional conditions, which constitutes a turbulence closure:

$$K_m=\frac{T}{L}\left(2l^2\left|\frac{\partial\sqrt{U^2+V^2}}{\partial z}\right|+\frac{\nu}{L}\right), \tag{6}$$

where $\nu=10^{-5}$ m^2/s is the molecular kinematics viscosity, $l=\kappa z$ is a mixing length scale in the simplest case, $\kappa=0.39$ is von Karman constant and $L=890$ m is the depth of the turbulent layer.

3. Problem solution

To solve the problem (1)-(6) we will introduce the following two functions:

$$u(U,V)=\int_{U_0(0)}^{U} K_m dU_1,\quad \vartheta(U,V)=\int_{V_0(0)}^{V} K_m dV_1.$$

Since $\dfrac{du(U,V)}{dU}=K_m(U,V)$, $\dfrac{\partial u(U,V)}{\partial t}=K_m(U,V)\dfrac{\partial U(t,z)}{\partial t}$, $\dfrac{\partial u(U,V)}{\partial z}=K_m(U,V)\dfrac{\partial U(t,z)}{\partial z}$,
then the equations become

$$\frac{\partial u(t,z)}{\partial t}=K_m(u,\vartheta)\frac{\partial^2 u(t,z)}{\partial z^2}+F_1(\vartheta), \tag{7}$$

$$\frac{\partial \vartheta(t,z)}{\partial t}=K_m(u,\vartheta)\frac{\partial^2 \vartheta(t,z)}{\partial z^2}+F_2(u), \tag{8}$$

where

$$F_1(\vartheta)\overset{def}{\equiv}Tf\cos(\alpha)(\vartheta-V_g),\quad F_1(u)\overset{def}{\equiv}Tf\cos(\alpha)(U_g-u).$$

Having introduced the Varshavsky integral transformation (for examples, see [6]) $h^{(u)}(u,\vartheta)=\int_0^u \frac{1}{K_m}du_1$, we obtain

$$h^{(u)}(u,\vartheta)=\int_0^u \frac{1}{K_m}du_1=\int_0^u \frac{2\sqrt{u_1^2+\vartheta^2}}{\frac{2l^2T}{L}\left(u_1\frac{\partial u_1}{\partial z}+\vartheta\frac{\partial\vartheta}{\partial z}\right)+\frac{T\nu}{L^2}\sqrt{u_1^2+\vartheta^2}}du_1=\frac{2L^2}{T\nu}\int_0^u \frac{du_1}{1+\frac{2l^2L}{\nu}\frac{u_1\frac{\partial u_1}{\partial z}+\vartheta\frac{\partial\vartheta}{\partial z}}{\sqrt{u_1^2+\vartheta^2}}}=$$

$$=\frac{2L^2}{T\nu}\left\{\frac{1}{2}\left(1+\frac{u^2}{2}\right)\left(1+\frac{u\vartheta^2}{2}\right)+u^2\vartheta+\left(\frac{2l^2T}{L}+\vartheta\right)u\right\}.$$

Similarly, if we consider the Varshavsky integral transformation $h^{(\vartheta)}(u,\vartheta)=\int_0^{\vartheta}\frac{1}{K_m}d\vartheta_1$, then we have

$$h^{(\vartheta)}(u,\vartheta)=\int_0^{\vartheta}\frac{1}{K_m}d\vartheta_1=\frac{2L^2}{T\nu}\left\{\frac{1}{2}\left(1+\frac{\vartheta^2}{2}\right)\left(1+\frac{\vartheta u^2}{2}\right)+\vartheta^2 u+\left(\frac{2l^2T}{L}+u\right)\vartheta\right\}.$$

Now in order to make use Biot variational principle (see [7]) we will introduce and calculate the following functions:

$$F^{(u)}(u,\vartheta)=\int_0^u \frac{u_1}{K_m}du_1=\frac{2L^2}{T\nu}\left\{\frac{\left(\frac{2l^2T}{L}+\vartheta\right)^2 u}{2}+\frac{1}{6}\left(1+\frac{u^2}{2}\right)\left(2+\frac{u^3\vartheta}{2}\right)+\frac{u^2\vartheta^2}{2}\right\},$$

$$F^{(\vartheta)}(u,\vartheta)=\int_0^{\vartheta} \frac{\vartheta_1}{K_m}d\vartheta_1=\frac{2L^2}{T\nu}\left\{\frac{\left(\frac{2l^2T}{L}+u\right)^2 \vartheta}{2}+\frac{1}{6}\left(1+\frac{\vartheta^2}{2}\right)\left(2+\frac{\vartheta^3 u}{2}\right)+\frac{u^2\vartheta^2}{2}\right\},$$

$$V^{(u)}\left((u,\vartheta)=q_1\right)=\int_0^{q_1(t)}F^{(u_1)}(u_1,\vartheta)du_1=\int_0^{q_1(t)}du_1\int_0^{u_1}\frac{u_2}{K_m(u_2,\vartheta)}du_2,$$

$$V^{(\vartheta)}\left((u,\vartheta)=q_2\right)=\int_0^{q_2(t)}F^{(\vartheta_1)}(u,\vartheta_1)d\vartheta_1=\int_0^{q_2(t)}d\vartheta_1\int_0^{\vartheta_1}\frac{\vartheta_2}{K_m(u,\vartheta_2)}d\vartheta_2,$$

where

$$u = c_1\left(1-\frac{z}{q_1(t)}\right)^2 + F_1,\ c_1 = const,\ \ \vartheta = c_2\left(1-\frac{z}{q_2(t)}\right)^2 + F_2,\ c_2 = const.$$

After calculations of integrals in the expressions for the introduced functions $V^{(u)}(q_1)$ and $V^{(\vartheta)}(q_2)$ we obtain that

$$V^{(u)}(u,\vartheta) = \frac{7}{61}c_1^2 q_1 - \frac{1}{3}q_1^2 q_2 f\cos(\alpha)V_g, \tag{9}$$

$$V^{(\vartheta)}(u,\vartheta) = \frac{7}{61}c_2^2 q_2 + \frac{1}{3}q_1 q_2^2 f\cos(\alpha)U_g. \tag{10}$$

Now we can consider the following integrals and calculate their: $H^{(u)}(q_1) = \int_z^{q_1} h^{(u)}dz$ and $H^{(\vartheta)}(q_2) = \int_z^{q_2} h^{(\vartheta)}dz$. Really, having designated $\xi = 1-\frac{z}{q_1}$ and $\eta = 1-\frac{z}{q_2}$ we can write

$$H^{(u)}(\xi,\eta) = q_1\int_0^{\xi} h^{(u)}d\xi = \frac{2L^2}{T\nu}\left\{\frac{4Tl^2}{3}\xi^2\eta + \frac{1}{40}\xi^5 + \frac{1}{10}\eta^3\right\}c_1 q_1,$$

$$H^{(\vartheta)}(\xi,\eta) = q_2\int_0^{\eta} h^{(\vartheta)}d\eta = \frac{2L^2}{T\nu}\left\{\frac{4Tl^2}{3}\xi\eta^2 + \frac{1}{40}\eta^5 + \frac{1}{10}\xi^3\right\}c_2 q_2.$$

It follows that

$$D^{(u)}(q_1) \overset{def}{\equiv} \frac{1}{2}\int_0^{q_1}\left(\frac{\partial H^{(u)}(q_1)}{\partial t}\right)^2 dz = \frac{12l^2L}{\nu}q_1^2(q_1')^3 + \frac{1}{8}q_1(q_1')^2 + \frac{1}{2}c_1 q_1, \tag{11}$$

$$D^{(\vartheta)}(q_2) \overset{def}{\equiv} \frac{1}{2}\int_0^{q_2}\left(\frac{\partial H^{(\vartheta)}(q_2)}{\partial t}\right)^2 dz = \frac{12l^2L}{\nu}q_2^2(q_2')^3 + \frac{1}{8}q_1(q_2')^2 + \frac{1}{2}c_2 q_2, \tag{12}$$

Now from Biot variational principle, we can write the following two Lagrange-Biot equations ([7]):

$$\frac{\partial V^{(u)}}{\partial q_1} + \frac{\partial D^{(u)}}{\partial q_1'} = const, \tag{13}$$

$$\frac{\partial V^{(\vartheta)}}{\partial q_1} + \frac{\partial D^{(\vartheta)}}{\partial q_1'} = const. \tag{14}$$

Substituting the relevant expressions for $V^{(u)}$, $V^{(\vartheta)}$, $D^{(u)}$, $D^{(\vartheta)}$ from (9)-(12) in (13) and (14) we obtain the following system of two ODE:

$$\frac{7}{61}c_1^2 + \frac{2}{3}q_1 q_2 fT\sin(\alpha)V_g + \frac{36l^2L}{5\nu}q_1^2(q_1')^2 q_1'' + \frac{1}{2}q_1 q_1' q_1'' = 0, \tag{15}$$

$$\frac{7}{61}c_4^2 + \frac{2}{3}q_1 q_2 fT\sin(\alpha)U_g + \frac{36l^2L}{5\nu}q_2^2(q_2')^2 q_2'' + \frac{1}{2}q_2 q_2' q_2'' = 0, \tag{16}$$

Let we have determined the analytical solution $\{q_1^{sol.}(t), q_1^{sol.}(t)\}$ of the system (15)-(16). Then the solution $\{u(t,z), \vartheta(t,z)\}$ of the reduced problem (7)-(8) is the functions

$$u(t,z) = c_1\left(1-\frac{z}{q_1^{sol.}(t)}\right)^2 + fT\cos(\alpha)\left(q_1^{sol.}(t) - V_g\right), \tag{17}$$

$$\vartheta(t,z) = c_2\left(1-\frac{z}{q_2^{sol.}(t)}\right)^2 + fT\cos(\alpha)\left(U_g - q_2^{sol.}(t)\right). \tag{18}$$

Here constants c_1 and c_2 can be found from the initial and boundary conditions (3)-(5).

In order to investigate features of proposed solutions for U and V we look here only of begining stage of the boundary layer formation, when time t is relatively small. For this case the system of equation (15)-(16) for $q_1(t)$ and $q_2(t)$ can be solved by using asymptotic expansion around $t=0$. So $q_1(t)$ and $q_2(t)$ can be written as:

$$q_1(t) = A_0 + A_1 t + A_2 t^2, \tag{19}$$

$$q_2(t) = B_0 + B_1 t + B_2 t^2, \tag{20}$$

assuming higher order coefficients in this expansion are small and can be omitted.

Coefficients A_0 and B_0 can be resolved from (19)-(20) taking into account that boundary conditions $U(0,0)=0$ and $V(0,0)=0$, so obtaining as the result that $u(0,0)=0$ and $\vartheta(0,0)=0$ too. Namely, for A_0 and B_0 we have $A_0 = \dfrac{c_2 + U_g Tf\cos(\alpha)}{Tf\cos(\alpha)}$, $B_0 = \dfrac{V_g Tf\cos(\alpha) - c_1}{Tf\cos(\alpha)}$.

After substitution $q_1(t)$ and $q_2(t)$ from (19)-(20) into the (15)-(16) we obtain nonlinear system for four equations which should be solved in order to find A_1, A_2, B_1 and B_2. This system reads:

$$A_0 - A_0 A_1 A_2 + \frac{72 l^2 L}{5\nu} A_0^2 A_1^2 A_2 - \frac{7}{61} c_1^2 - \frac{2}{3} V_g Tf\cos(\alpha) A_0 B_0 = 0, \tag{21}$$

$$A_1 - A_1^2 A_2 - 2A_0 A_2^2 + \frac{144 l^2 L}{5\nu} A_0 A_1^3 A_2 - \frac{288 l^2 L}{5\nu} A_0^2 A_1 A_2^2 - \frac{2}{3} V_g Tf\cos(\alpha)\left(A_1 B_0 + A_0 B_1\right) = 0, \tag{22}$$

$$B_1 - B_1^2 B_2 - 2B_0 B_2^2 + \frac{144 l^2 L}{5\nu} B_0 B_1^3 B_2 - \frac{288 l^2 L}{5\nu} B_0^2 B_1 B_2^2 - \frac{2}{3} U_g Tf\cos(\alpha)\left(A_1 B_0 + A_0 B_1\right) = 0, \tag{23}$$

$$2B_2^3 + \frac{432 l^2 L}{5\nu} B_1^3 B_2^2 + \frac{1152}{5\nu} B_0 B_1 B_2^3 + \frac{2}{3} U_g Tf\cos(\alpha)\left(A_2 B_1 + A_1 B_2\right) = 0. \tag{24}$$

In the considered case, when $V_g = 0$, the system (21)-(24) splits into two independent ones in respect of coefficients A and B. Moreover, to obtain real solutions of the systems for A and B the absolute values of the coefficients c_1 and c_2 should be equal, $|c_1| = |c_2|$. Here for calculations we use the following values of $c_1 = 15.8$ and $c_2 = -15.8$.

System (21)-(24) defines relations $U = U(u,\vartheta)$ and $V = V(u,\vartheta)$. To resolve these relations we rewrite (17)-(18) in the following form by subdivided all integration regions into sufficiently small parts and provided in each part a respective integration:

$$u(U,V) = \frac{2Tl^2}{L}\sum_i \int_{U_i}^{U_{i+1}} x^2 \frac{\partial}{\partial x}\left(U_1^2 + V^2\right)^{\frac{1}{2}} dU_1 + \frac{T\nu}{L^2} U, \tag{25}$$

$$\vartheta(U,V) = \frac{2Tl^2}{L}\sum_i \int_{V_i}^{V_{i+1}} x^2 \frac{\partial}{\partial x}\left(U + V_1^2\right)^{\frac{1}{2}} dV_1 + \frac{T\nu}{L^2} V, \tag{26}$$

As the next, we use a following equivalent formulation:

$$\frac{\partial}{\partial x}\left(U^2 + V^2\right)^{\frac{1}{2}} = \frac{\left(U^2 + V^2\right)^{\frac{1}{2}}}{2} \frac{\partial}{\partial x}\ln\left(U^2 + V^2\right). \tag{27}$$

By using the Bonnet's second mean value theorem and formula (27), the system (25)-(26) can be rewritten in the following form:

$$u(U,V) = \frac{Tl^2}{L}\sum_i x_i^2 \frac{\partial}{\partial x}\ln\left(U^2 + V^2\right)_i^{\frac{1}{2}} \int_{U_i}^{U_{\xi i}} \left(U_1^2 + V^2\right)^{\frac{1}{2}} dU_1 + \frac{T\nu}{L^2} U, \quad U_i < U_{\xi_i} < U_{i+1}, \tag{28}$$

$$\vartheta(U,V) = \frac{Tl^2}{L}\sum_i x_i^2 \frac{\partial}{\partial x}\ln\left(U^2 + V^2\right)_i^{\frac{1}{2}} \int_{V_i}^{V_{\xi i}} \left(U^2 + V_1^2\right)^{\frac{1}{2}} dV_1 + \frac{T\nu}{L^2} V, \quad V_i < V_{\xi_i} < V_{i+1}, \tag{29}$$

Providing here iteratively numerical calculations of (28)-(29) we approximately assumed $U_{\xi_i} = U_{i+1}$ and $V_{\xi_i} = V_{i+1}$. Results of numerical calculations are shown on Fig.1.

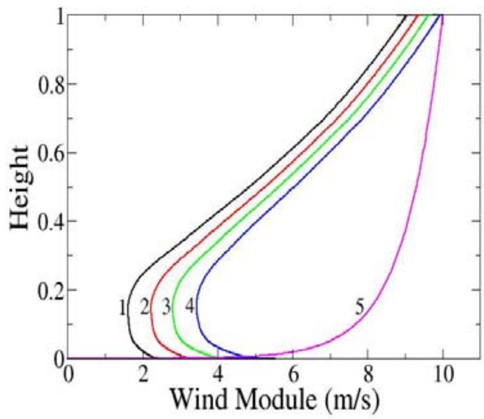

Fig.1. Wind module distributions in the different time moments: $1 - t = 0.0125$, $2 - t = 0.0250$, $3 - t = 0.0375$, $4 - t = 0.005$, $5 - t = 1$, *long time calculations (quasi steady-state solution) using propagator scheme.*

Calculations of long time processes are provided by using propagator difference scheme, see Fig.1. It is shown in [5], that stability restrictions for the propagator scheme become more weaker in comparison to traditional semi-implicit difference schemes. In [5] it is proven that the scheme is unconditionally monotonic, it has truncation errors of the first order in time and of the second order in space. Propagator scheme is adopted for solution of problem (1)-(6) due to low order truncation error does not reflect the boundary layer formation in details. In Fig.1 only long time calculations (quasi steady-state solution) for wind module distribution are shown. Although, it should be noted that after properly chosen space grid mean values of von Karman constant and friction velocity, numerically calculated by using propagator scheme, can be obtained close to realistic. This allows considering that higher order propagator difference scheme can improve resolution in time and space, and will be more adopted for boundary layer calculations.

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