ON PROBLEM OF CONTROL BY SYSTEMS WITH CONCENTRATED PARAMETERS ON SPECIAL CLASS OF CONTROL FUNCTIONS

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In controlling numerous real-world processes, realization of partial variations in the values of the control actions is either associated with difficulties of realization or cannot in general be achieved. From the practical point of view, therefore, there is a need to investigate problems of optimal control in the class of given control actions which realization does not cause technical difficulties. In the present article nonlinear problems of optimal control of processes that can be described by systems of ordinary differential equations are investigated. In these problems the coefficients participating in the formula for the controls are parameters which must be optimized and, what is most important, constancy intervals of these coefficients must be optimized.

A method of numerical determination of the optimal values of the coefficients participating in the formula for the controls and constancy intervals of these coefficients based on first-order finite-dimensional optimization methods and formulas for the gradient of a target functional with respect to the optimized parameters is proposed.

We consider the problem of optimal control by objects described by systems of nonlinear differential equations. Suppose that the state of a controlled object is described by the following Cauchy problem:

$$\dot{x}(t) = f(x,u,t),\ t \in (0,T], \qquad x(0) = \bar{x}_0, \tag{1}$$

where $x = x(t) \in E^n, t \in [0,T]$ – is the phase state of the object; the control $u(t) \in E^r, t \in [0,T]$ is determined by decomposition on the given functions $\varphi_m(t-\tau_{j-1}),\ m = \overline{1,M},\ t \in [\tau_{j-1},\tau_j)$ with unknown optimized coefficients on each half-interval $[\tau_{j-1},\tau_j)$, obtained by partitioning the segment $[0,T]$ at $(L-1)$ optimized points $\tau_j,\ j = 1,...,L$, i.e.

$$u(t) = \sum_{m=1}^{M} c_m^j \varphi_m(t-\tau_{j-1}),\ t \in [\tau_{j-1},\tau_j),\ c_m^j \in E^r, m = \overline{1,M}, \tag{2}$$

$$\tau_{j-1} \le \tau_j, \quad j = 1,...,L, \qquad \tau_0 = 0, \ \ \tau_L = T \tag{3}$$

and the admissible values of the control belong to some set U, in particular, to the following parallelepiped:

$$U = \{u : u = u(t),\ \alpha \le u(t) \le \beta,\ \alpha,\beta \in E^r, t \in [0,T]\}. \tag{4}$$

The problem under conditions (1)-(4) consists in finding vectors $C = (C^1,...,C^L) = (c_1^1,...,c_M^1,...,c_1^L,...,c_M^L)$ and $\tau = (\tau_1,...,\tau_{L-1})$ such that the given functional

$$J(u) = \bar{J}(C,\tau) = \int_0^T f^0(x,u,t)\,dt + \Phi(x(T)) \tag{5}$$

takes on its minimal value. It is assumed that the given functions f^0, Φ, $\varphi_m(t-\tau_{j-1}), m = \overline{1,M}, t \in [\tau_{j-1},\tau_j)$ and the vector function f together with the partial derivatives are continuous with respect to (x,u).

For numerical solving the problem (1)-(5) we use the scheme suggested in [1-3]. For this purpose we introduce on the segment $[0,T]$ the uniform lattice region

$$\Omega = \{ t_i : t_i = ih, \, i = 0,...,N, \, h = T/N \}.$$

Here N is a given natural number. Without any loss in generality, to simplify the computational formulas of the system (1) we will accomplish the approximation by means of the explicit Euler technique, and will approximate the integral takes part occurs in the expression of the functional (5) by the method of rectangles. As a result, we obtain the following finite-dimensional mathematical programming problem:

$$x_{i+1} = x_i + hf(x_i, u_i, t_i), \; i = 0,...,N-1, \qquad x_0 = \bar{x}_0, \tag{6}$$

$$u_i = \begin{cases} \sum_{m=1}^{M} c_m^j \varphi_m^{i,j-1}, & [t_i, t_{i+1}) \subset [\tau_{j-1}, \tau_j), \; j = \overline{1, L}, \\ \frac{1}{h}\left[(t_{i+1} - \tau_j) \sum_{m=1}^{M} c_m^{j+1} \varphi_m^{i+1,j} + (\tau_j - t_i) \sum_{m=1}^{M} c_m^j \varphi_m^{i,j-1} \right], & \tau_j \in [t_i, t_{i+1}), \; j = \overline{1, L-1}, \end{cases} \tag{7}$$

$$i = \overline{0, N-1},$$

$$I(C, \tau) = \Phi(x_N) + h \sum_{i=0}^{N-1} f^0(x_i, u_i, t_i) \to \min_{C,\tau}, \tag{8}$$

taking into account conditions (3), (4). Here $\varphi_m^{i,j} = \varphi_m(t_i - \tau_j)$, $m = \overline{1, M}$, $i = \overline{0, N-1}$, $j = \overline{0, L-1}$.

From (7) it is evident that if the switching time τ_j of the control lies between the nodal points t_i and t_{i+1}, the value of the control u_i will be approximated by a linear combination of the values of $\sum_{m=1}^{M} c_m^j \varphi_m^{i,j-1}$ and $\sum_{m=1}^{M} c_m^{j+1} \varphi_m^{i+1,j}$.

To solve problem (6)-(8), that is, to determine the optimal values of the vectors C and τ, we will use numerical first-order finite-dimensional optimization methods, in particular, the iteration method of projection of the gradient of the functional in the space of optimized parameters (C, τ):

$$(C^{k+1}, \tau^{k+1}) = P_{(3),(4)}\left[(C^k, \tau^k) - \alpha \left(dI(C^k, \tau^k)/dC, \, dI(C^k, \tau^k)/d\tau \right) \right], k = 0,1,..., \tag{9}$$

where $P_{(3),(4)}$ is the operator for projection of the vector (C, τ) to an admissible region of parameters determined by the constraints (3) and (4), and (C^0, τ^0) is some specified initial approximation for the optimized parameters; the vectors

$$\begin{aligned} dI/d\tau &= (dI/d\tau_1, ..., dI/d\tau_{L-1})^*, \\ dI/dC &= (dI/dc_1^1, ..., dI/dc_M^1, ..., dI/dc_1^L, ..., dI/dc_M^L)^* \end{aligned} \tag{10}$$

determine the gradient of the functional of problem (6)–(8), the formulas for the components of which will be obtained below; * is transposition sign.

We introduce the following vectors of the impulse variables ([1]):

$$p_i = dI/dx_i, \quad p_i \in E^n, \quad i = 0,...,N.$$

Here the derivative is understood as a total derivative, taking into account interdependence of the values of x_i, $i = 0,...,N$, from (6). Thus, considering (6), it follows that

$$p_i = h \frac{\partial f^0(x_i, u_i, t_i)}{\partial x_i} + \left[E + h \frac{\partial f^{\mathrm{T}}(x_i, u_i, t_i)}{\partial x_i} \right] p_{i+1}, \; i = N-1,...,0, \quad p_N = \frac{\partial \Phi(x_N)}{\partial x_N}, \tag{11}$$

where E is an n-dimensional unit matrix. The system (11) will be called the adjoint system.

Let us suppose that the switching time τ_j lies between the nodal points $t_{k_{j-1}}$ and t_{k_j}, that is, $\tau_j \in \left[t_{k_{j-1}}, t_{k_j}\right)$, $j = 1,...,L-1$. Then the components of the gradient dI/dc_m^j, $m = 1,...,M$, $j = \overline{1,L}$, are determined as follows:

$$\frac{dI}{dc_m^j} = \frac{\partial I}{\partial c_m^j} + \sum_{s=k_{j-1}}^{k_j} \frac{\partial x_s}{\partial c_m^j} p_s = h \sum_{s=k_{j-1}}^{k_j} \left[\frac{\partial f^0(x_{s-1}, u_{s-1}, t_{s-1})}{\partial u_{s-1}} \frac{\partial u_{s-1}}{\partial c_m^j} \right] +$$
$$+ h \sum_{s=k_{j-1}}^{k_j} \left[\frac{\partial f^*(x_{s-1}, u_{s-1}, t_{s-1})}{\partial u_{s-1}} \frac{\partial u_{s-1}}{\partial c_m^j} p_s \right] = \tag{12}$$
$$= h \sum_{s=k_{j-1}}^{k_j} \left[\frac{\partial f^0(x_{s-1}, u_{s-1}, t_{s-1})}{\partial u_{s-1}} \frac{\partial u_{s-1}}{\partial c_m^j} + \frac{\partial f^*(x_{s-1}, u_{s-1}, t_{s-1})}{\partial u_{s-1}} \frac{\partial u_{s-1}}{\partial c_m^j} p_s \right],$$
$$j = \overline{1,L}, \quad m = \overline{1,M},$$

and the partial derivatives $\partial u_s / \partial c_m^j$, $s = \overline{k_{j-1}-1, k_j-1}$, $j = \overline{1,L}$, $m = \overline{1,M}$ are determined from (7):

$$\frac{\partial u_s}{\partial c_m^j} = \begin{cases} \varphi_m^{s,j-1}, & s = k_{j-1},...,k_j-2, \\ (t_{s+1} - \tau_{j-1}) \varphi_m^{s+1,j-1} / h, & s = k_{j-1}-1, \\ (\tau_j - t_s) \varphi_m^{s,j-1} / h, & s = k_j - 1. \end{cases} \tag{13}$$

For the components of the gradient $dI/d\tau_j$, $j = \overline{1,L-1}$, we have

$$\frac{dI}{d\tau_j} = \frac{\partial I}{\partial \tau_j} + \frac{\partial x_{k_j}}{\partial \tau_j} p_{k_j} = \frac{\partial I}{\partial \tau_j} + \frac{\partial x_{k_j}}{\partial u_{k_j-1}} \frac{\partial u_{k_j-1}}{\partial \tau_j} p_{k_j} =$$
$$= h \frac{\partial u_{k_j-1}}{\partial \tau_j} \left[\frac{\partial f^0\left(x_{k_j-1}, u_{k_j-1}, t_{k_j-1}\right)}{\partial u_{k_j-1}} + \frac{\partial f^*\left(x_{k_j-1}, u_{k_j-1}, t_{k_j-1}\right)}{\partial u_{k_j-1}} p_{k_j} \right]. \tag{14}$$

The partial derivatives $\partial u_{k_j-1} / \partial \tau_j$, $j = \overline{1,L}$, are determined directly from (7):

$$\frac{\partial u_{k_j-1}}{\partial \tau_j} = \frac{1}{h} \left[\sum_{m=1}^{M} c_m^j \varphi_m^{k_j-1,j-1} - \sum_{m=1}^{M} c_m^{j+1} \varphi_m^{k_j,j} \right] = \frac{1}{h} \sum_{m=1}^{M} \left(c_m^j \varphi_m^{k_j-1,j-1} - c_m^{j+1} \varphi_m^{k_j,j} \right). \tag{15}$$

Then from (14) and (15) we find for $j = 1,...,L-1$,

$$\frac{dI}{d\tau_j} = \left[\frac{\partial f^0\left(x_{k_j-1}, u_{k_j-1}, t_{k_j-1}\right)}{\partial u_{k_j-1}} + \frac{\partial f^*\left(x_{k_j-1}, u_{k_j-1}, t_{k_j-1}\right)}{\partial u_{k_j-1}} p_{k_j} \right] \times$$
$$\times \sum_{m=1}^{M} \left(c_m^j \varphi_m^{k_j-1,j-1} - c_m^{j+1} \varphi_m^{k_j,j} \right). \tag{16}$$

Formulas (12) and (16) determine the components of the gradient of the functional (8) of problem (6)-(8). A realization of the iterative process (9) consists in the following stages:

1) a solution of the approximate Catchy problem $x_i \in E^n$, $i = 0,...,N$, is found from formulas (6), (7) for the current values of the vector $\left(C^k, \tau^k\right) \in E^{L(M+1)-1}$;

2) from formulas (13), (14) vectors of impulses $p_i \in E^n$ are determined in reverse order, beginning with $i = N$ to $i = 0$;

3) from formulas (12), (16) the components of the vector of the gradient (8) are determined;

4) the procedure (9) is accomplished with a choice of α from the condition of one-dimensional minimization of the functional (8) (in view of the "simplicity" of the admissible region of the parameters (3), (4), the operation of projection $P_{(3),(4)}$ does not represent any difficulty), and a new approximation $\left(C^{k+1}, \tau^{k+1}\right)$ is determined. In the case when the optimality condition is not satisfied or when the iteration process halts, stages 1-4 are repeated.

Remark. It is clear that the choice of the technique of the Euler method for the approximation of problem (1)-(5) is not of essential importance for the proposed approach. The formulas that have been obtained here may be easily extended to other techniques of discretization of the initial problem.

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