The Second International Conference "Problems of Cybernetics and Informatics"
September 10-12, 2008, Baku, Azerbaijan. Section #5 "Control and Optimization"
www.pci2008.science.az/5/11.pdf

ON IMPULSIVE OPTIMAL CONTROL PROBLEM FOR SYSTEMS WITH CONCENTRATED PARAMETERS

Yegana Ashrafova

Cybernetics Institute of ANAS, Baku, Azerbaijan, *y_aspirant@yahoo.com*

Many-branched applications to dynamic of cosmic apparatus, economy, ecology, quantum electronic, robot technique were stimulated interest to none-classic optimal control problems, where the trajectories of dynamic systems can be discontinuous, and controls include impulses – momentary influences on system cumulative (stroke) character [1, 2].

One of the major problems in the field of optimization of systems with impulsive controls is the problem of obtaining constructive conditions for optimality, including the specificity of this class of systems, and giving opportunity to use them in order to solve the problem numerically.

The necessary conditions for optimality of the impulsive regimes in the maximum principle form for different kind of systems and the way of determining impulsive regimes as partial case of the generalized functions were obtained in [3-5]. But these formulas aren't quietly for using in the numerical calculations. The constructive formulas allowing using the optimization methods of the first order were obtained in this work when the number of impulsive influences is given.

An impulsive optimal control problem by the objects described by system of ordinary differential equations is considered. Particularly, let us consider the problem of minimization of the functional

$$J(u) = I(q,\theta) = \alpha_1 \int_0^T f^0(x,t)dt + \alpha_2 \Phi_1(x(T)) + \alpha_3 \Phi_2(q,\theta), \tag{1}$$

with conditions

$$\dot{x}_i(t) = f_i(x(t),t) + \sum_{j=1}^{L} b_{ij}(t) u_{ij}(t),\ 0 < t \le T,\ x_i(0) = \overline{x}_{0i},\ i = \overline{1,n} \tag{2}$$

$$u_{ij}(t) = q_{ij}\delta(t-\theta_j),\ i = 1,...,n,\ \ j = 1,..,L \tag{3}$$

Here $x(t) = (x_1(t),...,x_n(t))$ phase vector; $\delta(\cdot)-$is a Dirac function, mappings $f^0(x,t),\ f(x,t) = (f^1(x,t),..., f^n(x,t)),(b_1(t),...,b_L(t)),\quad \Phi_1(x),\Phi_2(q,\theta)$ of variables $(x,q,\theta,t) \in E^n \times E^L \times E^L \times [0,T]$ are given and it is assumed to be continuous along with its derivatives on all the arguments. The control influences:

$$u = (q,\theta) \in E^{(n+1)L},\quad q_i = (q_{i1},...,q_{iL}),\ \ i = \overline{1,n},\ \theta = (\theta_1,...,\theta_L)$$

satisfy the following conditions:

$$\sum_{i=1}^{n}\sum_{j=1}^{L} q_{ij}^2 \le Q\ ,\ \underline{q_j} \le q_{ij} \le \overline{q_j},\quad 0 \le \xi < \theta_j - \theta_{j-1} \le \eta,\quad \theta_j \in [0,T],\ i = \overline{1,n},\ j = \overline{1,L}\,. \tag{4}$$

Here $Q, \underline{q_j}, \overline{q_j}, \xi, \eta, L, \alpha_1, \alpha_2, T$ and initial vector x_0 is given.

With purpose of applying finite-dimensional optimization methods of the first order for the determining of optimal vector (q^*,θ^*), it will be obtained analytical formulas for the gradient of the functional (1): $gradJ(u) = \left(I'_q(q,\theta), I'_\theta(q,\theta)\right)$.

Let us consider the following Hamilton-Pontryagin function and adjoint system for the problem of (1)-(4):

$$H(\psi,x,u,t) = -\alpha_1 f_0(x,t) + \sum_{i=1}^{n}\psi_i(t) f_i(x,t) + \sum_{i=1}^{n}\sum_{j=1}^{L}\psi_i(t)b_{ij}(t)q_{ij}\delta(t-\theta_j),$$

$$\dot\psi_i(t) = \alpha_1 \frac{\partial f_0(x(t),t)}{\partial x_i} - \sum_{k=1}^{n}\psi_k(t)\frac{\partial f_k(x(t),t)}{\partial x_i},$$

$$\psi_i(T) = -\alpha_2 \frac{\partial \Phi_1(x(T))}{\partial x_i(T)}, i=\overline{1,n}, \qquad (6)$$

where $\psi(t)\in E^n$. The increment of the functional (1) for admissible control $u=u(t)$ and $u+\Delta u = u(t)+\Delta u(t)$ can be written in the following way [5]:

$$\Delta J(u) = \Delta I(q,\theta) = J(u+\Delta u) - J(u) = -\sum_{i=1}^{n}\sum_{j=1}^{L}\int_0^T \psi_i(t)b_{ij}(t)\Delta u dt + $$
$$+\alpha_3(\Phi_2(u+\Delta u)-\Phi_2(u)) + o(\|\Delta u\|). \qquad (7)$$

For the first, let us obtain the expression for $\frac{dI(q,\theta)}{dq_{ij}}, i=\overline{1,n},\ j=\overline{1,L}$. Let us write the functional increment in the following way:

$$\Delta_{q_{ij}} J(u) = -\int_0^T \psi_i(t)b_{ij}(t)\Delta q_{ij}\delta(t-\theta_j)dt + \alpha_3(\Phi_2(q+\Delta q,\theta)-\Phi_2(q,\theta)) + o(\|\Delta q_{ij}\|) = $$
$$= -\psi_i(\theta_j)b_{ij}(\theta_j)\Delta q_{ij} + \alpha_3(\Phi_2(q+\Delta q,\theta)-\Phi_2(q,\theta)) + o(\|\Delta q_{ij}\|).$$

Dividing both parts by Δq_{ij} and according to $o(\|\Delta q\|)/\Delta q_{ij}\to 0$ and converging to limit at $\Delta q_{ij}\to 0$ we will get the next expression:

$$\frac{dI(q,\theta)}{dq_{ij}} = -\psi_i(\theta_j)b_{ij}(\theta_j) + \alpha_3\frac{\partial\Phi_2(q,\theta)}{\partial q_{ij}},\ i=\overline{1,n},\ j=\overline{1,L}. \qquad (8)$$

And now let us get the expression for $\frac{dI(q,\theta)}{d\theta_j},\ j=\overline{1,L}$ For this case we will use the next δ_ε-functions:

$$\delta_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon},\ t\in[\theta-\varepsilon,\theta], \\ 0,\ t\notin[\theta-\varepsilon,\theta], \end{cases}$$

where $\varepsilon>0-$ is a small parameter, it is obviously that, $\delta(t)=\lim_{\varepsilon\to 0}\delta_\varepsilon(t)$. Let us give θ the increment $\Delta\theta=\varepsilon$. Then δ_ε-function will get the following increment:

$$\Delta_\theta\delta_\varepsilon(t-\theta) = \begin{cases} 0, & t\notin[\theta-\varepsilon,\theta+\Delta\theta], \\ -1/\varepsilon, & t\in[\theta-\varepsilon,\theta+\Delta\theta-\varepsilon], \\ 1/\varepsilon, & t\in[\theta+\Delta\theta-\varepsilon,\theta+\Delta\theta]. \end{cases}$$

Then for increment of functional on increment $\Delta u=(0,\Delta\theta), \Delta\theta = \left(0,...,\Delta\theta_j,....,0\right)$ by using (7), we have:

$$\Delta_{\theta_j} J(u) = -\sum_{i=1}^{n}\int_0^T \psi_i(t)b_{ij}(t)q_{ij}\Delta_\theta\delta_\varepsilon(t-\theta_j)dt + \alpha_3(\Phi_2(u+\Delta u)-\Phi_2(u)) + o(\|\Delta u\|) =$$

$$= -\sum_{i=1}^{n}\frac{q_{ij}}{\varepsilon}\left(\int_{\theta_j-\varepsilon}^{\theta_j}\psi_i(t+\Delta\theta_j)b_{ij}(t+\Delta\theta_j) - \psi_i(t)b_{ij}(t)dt\right) + \alpha_3(\Phi_2(q,\theta+\Delta\theta)-\Phi_2(q,\theta)) + o(\|\Delta u\|),$$

By using the separating of the function $\psi_i(t)b_{ij}(t)$ in the Taylor rank around the point t, we receive the following:

$$\int_{\theta_j-\varepsilon}^{\theta_j}[\psi_i(t+\Delta\theta_j)b_{ij}(t+\Delta\theta_j)-\psi_i(t)b_{ij}(t)]dt=\int_{\theta_j-\varepsilon}^{\theta_j}[(\psi_i(t)(b_{ij}(t))'\Delta\theta_j+$$

$$+(\psi_i(t)(b_{ij}(t))''\frac{\Delta\theta_j^2}{2}+o(\Delta\theta_j^2)]dt=\Delta\theta_j\psi_i(t)(b_{ij}(t))\,|_{\theta_j-\varepsilon}^{\theta_j}+(\psi_i(t)b_{ij}(t))'\frac{\Delta\theta_j^2}{2}\,|_{\theta_j-\varepsilon}^{\theta_j}+o(\Delta\theta_j^2).$$

By noticing this in the formula of the increment of functional (7), dividing both part by $\Delta\theta_j$ and converging to limit at $\Delta\theta_j\to 0$, $\varepsilon\to 0$, we will achieve:

$$\frac{dI(q,\theta)}{d\theta_j}=-\sum_{i=1}^{n}q_{ij}(\lim_{\varepsilon\to 0}\lim_{\Delta\theta_j\to 0}\frac{\psi_i(\theta_j)b_{ij}(\theta_j)-\psi_i(\theta_j-\varepsilon)b_{ij}(\theta_j-\varepsilon)}{\varepsilon}+$$

$$+\lim_{\varepsilon\to 0}\lim_{\Delta\theta_j\to 0}\frac{\Delta\theta_j}{2}\frac{(\psi_i(\theta_j)b_{ij}(\theta_j))'-(\psi_i(\theta_j-\varepsilon)b_{ij}(\theta_j-\varepsilon))'}{\varepsilon})+\alpha_3\lim_{\Delta\theta_j\to 0}\frac{\Phi_2(q,\theta+\Delta\theta)-\Phi_2(q,\theta)}{\Delta\theta_j},$$

$$\frac{dI(q,\theta)}{d\theta_j}=-\sum_{i=1}^{n}q_{ij}(\psi_i(\theta_j)b_{ij}(\theta_j))'_{\theta_j}+\alpha_3\frac{\partial\Phi_2(q,\theta)}{\partial\theta_j},\quad j=\overline{1,L}. \tag{9}$$

It can be shown that at the case of $\varepsilon>\Delta\theta$ and $\varepsilon<\Delta\theta$ expression for components of functional gradient $\frac{dI(q,\theta)}{d\theta_j}$, $j=\overline{1,L}$ same as (9).Thus, the next theorem is proved.

Theorem: The components of the functional gradient on parameters of the control influences in the problem (1), (4) are determining by the formulas (8), (9).

Remark: If **a**n impulsive optimal control problem by the objects described by system of non-linear ordinary differential equations

$$\dot{x}_i(t)=f_i(x(t),t)+\sum_{j=1}^{L}f_{ij}(x(t),q_{ij},t)\delta(t-\theta_j),\ t\in(0,T],$$

$$x_i(0)=x_{0i},\ i=\overline{1,n},$$

then it can analogically to (8), (9) be obtained formulas for capacity and time components of impulsive influences in the following form:

$$\frac{dI(q,\theta)}{dq_{ij}}=-\frac{\partial f_{ij}(x(\theta_j),q_{ij},\theta_j)}{\partial q_{ij}}\psi_i(\theta_j)+\alpha_3\frac{\partial\Phi_2(q,\theta)}{\partial q_{ij}},i=\overline{1,n},\ j=\overline{1,L},$$

$$\frac{dI(q,\theta)}{d\theta_j}=-\sum_{i=1}^{n}q_{ij}\left(f_{ij}(x(\theta_j),q_{ij},\theta_j)\psi_i(\theta_j)\right)'_{\theta_j}+\alpha_3\frac{\partial\Phi_2(q,\theta)}{\partial\theta_j},\ j=\overline{1,L}.$$

Where $\psi(t)\in E^n$ is a solution of the adjoint system analogical to (6).

By using the receiving formulas for functional gradient, let us yield the results of applying them in the next test problem.

$$\dot{x}(t)=tx(t)+(t-1)q\delta(t-\theta),\ \ 0<t\le 1,\ \ \ x(0)=1,$$

$$J(u)=I(q,\theta)=\int_0^1 x^2(t)dt+0.4(q-8)^2+0.1(\theta-0.2)^2,0\le q\le 10$$

The accurate optimal control for this problem is unknown.

The gradient vector calculating by formulas (8), (9) at the arbitrary selected point (q,θ)=(6;0,7) is equal to: $gradJ(u)=\left(I'_q(q,\theta),I'_\theta(q,\theta)\right)$=(-1,4794;-4,6961).

The problem was numerically solved by using formulas (8), (9) with step $h=0.002$. The results of numerical experiments by using the method of the projection of adjoint gradients for different initial values of control vector with optimization accuracy $\varepsilon=0{,}001$ are shown on the table 1.

Table 1

The numerical results of the problem

(q^0,θ^0)	(q^*,θ^*)	J^0	J^*	The number of iter.
(6;0.7)	(8,000; 0,7779)	2,5639	1,0499	9
(4;0.6)	(8,000; 0,7799)	7,1852	1,0493	6
(2;0.9)	(7,959; 0,7839)	15,8539	1,0478	3
(2;0.8)	(8,000; 0,7799)	15,6761	1,0492	7
(8;0.2)	(7,968; 0,7779)	35,3643	1,0487	2
(4;0.5)	(8,000; 0,7819)	7,4929	1,0489	5

The constructive analytical formulas for the gradient of the target functional of the considered problem (when the number of impulses is given) is obtained which allow using first order optimization methods for solving the optimal control problem.

According to simplicity of realization of the impulsive control and their extensive using in technique, the suggested approach to constructing the strategy of control by the objects with concentrated parameters can find the wide application in the systems of control by those objects.

This work was supported by INTAS (project Ref. Nr 06-1000017-8909) in the frame of INTAS Collaborative Program with South Caucasian Republics 2006.

Literature

1. A. Bensusan, J.-L. Lions Impulsive control and kvazi-variasyon inequalities, (In Russian) M. Nauka, 1987. 600 p.
2. Dykhta V.A. Optimization dynamic systems with discontinuous trajectories and impulsive controls (In Russian) //Sorosov educational journal, №8, 1999.p.110-115.
3. Miller B.M. The conditions of optimality in problems of generalized controls I. The necessary conditions of optimality II. The sufficient conditions of optimality (In Russian) //Automation and telemechanics. 1992. №3 p. 50-58, №4, p. 39-48.
4. Miller B.M. The conditions of optimality in problem of control by system described by differential equation with measure (In Russian) //Automation and telemechanics 1982. №6. p.60-72.
5. Colocolnicova Q. A. The necessary conditions of second order of optimality special and impulsive regimes (In Russian) //Automation and telemechanics. 1990. №6. p.48-57.
6. Gabasov R., Kirillova F.M., The maximum principle in the theory of optimal control (In Russian) M., Nauka and technika, 1974, 272 p.